LOCAL ONE DIMENSIONAL DIFFERENCE SCHEMES ON NON-UNIFORM NETS*

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A number of economical difference schemes which are applicable to regions of a special form (parallelepipeds or regions formed from them, cf. [9]) have been put forward for the numerical solution of the linear heat conduction equation in several space variables ([1]-[9]). Equations with variable coefficients were studied in [7]-[12] and the other works are concerned with constant coefficients. In [10] we proposed a local one-dimensional method of variable directions for the linear equation (1.1_0) (cf. Section 1) and for the simplest quasi-linear equation (1.1_1), and this method was suitable for an arbitrary region in space G, on the boundary Γ of which boundary conditions of the first kind were given. We constructed a family of local one-dimensional schemes which were homogeneous with respect to space and cyclically homogeneous with respect to time. It was shown that all these schemes were absolutely stable with respect to the initial and the boundary data, and also with respect to the right-hand side of the equation (Theorem 1, [10]) and that they converged uniformly at a rate $O(h^2) + O(\tau)$, i.e. had the same order of accuracy as multi-dimensional implicit schemes (cf. [17]).

In this paper we consider local one-dimensional schemes on arbitrary non-uniform nets for linear and quasi-linear parabolic equations with the "heat conduction coefficient" $k_\alpha = k_\alpha(x, t, u)$ dependent on the "temperature" $u = u(x, t)$. The maximum principle, with the help of which the uniform regularity of the problem for the error $z = y - u$ with respect to the approximation error was established in [10], does not hold in the case $k_\alpha = k_\alpha(x, t, u)$ and in the case of non-uniform nets when $k_\alpha = k_\alpha(x, t)$ it enables us only to prove first order accuracy with respect to h (for problems II_0 and II_1, cf. Section 1). For a number of

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schemes II_k, $k = 1', 2', 3$ (cf. Section 1), the maximum principle has not been established. We shall therefore pay considerable attention (cf. Section 2) to the derivation of *a priori* estimates (estimates in the mean and uniform estimates) for very complex equations which are satisfied by the error $z = y - u$ where u is the solution of the differential equation and y is the solution of the corresponding difference problem. Since the coefficients of the space parts of the difference schemes satisfy only the boundedness condition, in order to derive *a priori* estimates we use the n-th rank energy inequality method developed in [14], [15], [17]. With this method we take into account the complex structure of the approximation error

$$\Psi_\alpha = \overset{\circ}{\psi}_\alpha + \psi_\alpha, \qquad \overset{\circ}{\psi}_\alpha = O(1), \quad \sum_{\alpha=1}^{p} \overset{\circ}{\psi}_\alpha = 0,$$

$$\psi_\alpha = \sum_{\alpha=1}^{p} [(\mu_{\alpha\beta})_{\hat{x}_\beta} + \psi^*_{\alpha\beta}], \qquad \mu_{\alpha\beta} = O(h_\beta^2), \qquad \psi^*_{\alpha\beta} = O(\hbar_\beta^2) + O(\tau).$$

Two methods have been suggested, in [10] and [12], for summing the principal part of the approximation error, $\overset{\circ}{\psi}_\alpha$. However, since the method of [10] should be used when the maximum principle holds and the method of [12] when $p = 2$ for a special form of the schemes, in Section 2 we suggest a third method which is very general and can be applied in all cases. The method for summing the local approximation errors with respect to space is based on methods from [13]-[18]. Together with the space net $\omega_h^{(1)}$, similar to the net in [10], we consider the net $\omega_h^{(2)}$, the boundary points of which are points of the boundary Γ of the region G and so the boundary conditions in them are given without extrapolation. It is shown in Section 3 that the order of accuracy on both nets is the same $(O(h^2))$. Of the results of Section 3 we note only one: if the scheme II_1 (cf. Section 1) converges uniformly on a uniform net at a rate $O(h^2) + O(\tau)$ then on an arbitrary non-uniform net it converges in the mean at the rate $O(\|h^2\| + \|\tau\|_0)$ and converges uniformly at the rate

$$O\left(\|h\|_0^2 \ln^\delta \frac{1}{H_*}\right) + O\left(\|\tau\|_0 \ln^\delta \frac{1}{H_*}\right),$$

where $\|h^2\|$ is the mean square value of $h^2 = \sum_{\alpha=1}^{p} h_\alpha^2$ on the net ω_h,

$\|h^2\|_0 = \max_{x \in \omega_h} h^2$, $\|\tau\|_0 = \max_{\omega_\tau} \tau_j$, $H_* = \min_{\omega_h} H$, $H = \prod_{\alpha=1}^{p} \hbar_\alpha$ is the volume of net meshes and $\delta > 1$ is an arbitrary number.

The *a priori* estimates we obtain enable us to prove without difficulty the convergence of all the local one-dimensional schemes of Section 1 in the case when the coefficients of the differential equation have discontinuities of the first kind on a finite number of hyperplanes parallel to the coordinate hyperplanes (cf. [17]. Choosing special sequences of nets $\omega_h(k)$, by analogy with [14], we can arrange that in this case our schemes will have second order accuracy with respect to h.

By concentrating the net ω_h near the boundary we can also attain second order accuracy for the third boundary problem in the case of the parallelepiped considered in [10], Section 1, Para. 7.

The methods of Section 2 enable us, in particular, to show that homogeneous difference schemes for elliptic equations as well as the splitting method of [7]-[9] for linear parabolic equations, retain their maximum order of accuracy on arbitrary non-uniform nets. A brief account of the corresponding results is given in Section 4.

1. Local one-dimensional schemes of variable direction on a non-uniform net

1. THE DIFFERENTIAL EQUATIONS

We shall consider the p-dimensional parabolic equations

$$c(x, t)\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} L_\alpha u + f, \tag{1}$$

where $x = (x_1, \ldots, x_\alpha, \ldots, x_p)$ is a point of p-dimensional Euclidean space R_p with the coordinates $x_1, \ldots, x_\alpha, \ldots, x_p$, $c(x, t) = c(x_1, \ldots, x_p, t)$. The differential operator $L_\alpha u$ and the function f are defined by one of the formulae:

$$L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x, t)\frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x, t)\frac{\partial u}{\partial x_\alpha}, \qquad f = f(x, t) - q(x, t)u, \tag{1_0}$$

$$L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x, t)\frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x, t, u)\frac{\partial u}{\partial x_\alpha}, \qquad f = f(x, t, u), \tag{1_1}$$

$$L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x, t, u)\frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x, t, u)\frac{\partial u}{\partial x_\alpha}, \quad f = f(x, t, u), \tag{1_2}$$

$$L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x, t, u)\frac{\partial u}{\partial x_\alpha}\right), \qquad f = f\left(x, t, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_p}\right). \tag{1_3}$$

Let G be an arbitrary p-dimensional finite region with boundary Γ, $\bar{G} = G + \Gamma$, $\bar{Q}_T = \bar{G} \times [0 \leqslant t \leqslant T]$, $Q_T = G \times (0 < t \leqslant T]$. In the cylinder $\bar{Q}_T$ we are required to find the solution $u = u(x, t)$ of the problem

$$\left.\begin{aligned} &c\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} L_\alpha u + f, \qquad (x, t) \in Q_T, \\ &u|_\Gamma = u_1(x, t), \quad x \in \Gamma, \quad t \in [0, T]; \qquad u(x, 0) = u_0(x), \quad x \in \bar{G}, \end{aligned}\right\} \tag{I}$$

where $u_0(x)$ and $u_1(x, t)$ are given functions.

Depending on the form of L_α of f we obtain problems I_0, I_1, I_2, I_3. In [10] we studied only the problems I_0 and I_1. The coefficients $k_\alpha(x, t, u)$, $c(x, t)$ satisfy the conditions

$$k_\alpha(x, t, u) \geqslant c_1 > 0, \quad c(x, t) \geqslant c_2 > 0, \quad \alpha = 1, \ldots, p,$$

where c_1 and c_2 are constants. By analogy with [16] we shall assume that

$$\partial k_\alpha / \partial u, \ \partial r_\alpha / \partial u, \ \partial f / \partial u, \ \partial f / \partial q_1, \ldots, \partial f / \partial q_p, \tag{2}$$

where $f = f(x, t, u, q_1, \ldots, q_p)$ are uniformly continuous in the whole region of variation of the arguments $(x, t) \in \bar{Q}_T$, u, $q_1, \ldots, q_p$. It is also assumed that each of the problems I_k, $k = 0, 1, 2, 3$, has a unique solution $u = u(x, t)$ continuous in $\bar{Q}_T$ and possessing all the derivatives which are required in the course of the argument. We use the same assumptions with respect to the region G as we used in [10]; the intersection of the region G by any straight line $\mathcal{L}_\alpha$, drawn through a point $x \in G$ parallel to the coordinate axis Ox_α consists of a finite number of intervals. It is not difficult to see that we can conduct our argument for the case when the straight line $\mathcal{L}_\alpha$ intersects Γ in two points only without loss of generality. Extension to the general case involves nothing more than further difficulty for the printer.

2. THE NETS

In [10] we studied schemes for the problems I_0 and I_1 on a uniform net. Here we shall consider two types of space nets, each of which is non-uniform.

Consider the set of nodes $x_i = (x_1^{(i_1)}, \ldots, x_\alpha^{(i_\alpha)}, \ldots, x_p^{(i_p)}) \in \bar{G}$ of a rectangular net covering $\bar{G}$, where $i_\alpha = 0, \pm 1, \pm 2, \ldots$, and let us construct two nets $\bar{\omega}_h^{(1)}$ and $\bar{\omega}_h^{(2)}$ on the region $\bar{G}$.

1. The net $\bar{\omega}_h^{(1)}$ is defined as in [10]. The nodes x_i and $x_{i'} = (x_1^{(i_1')}, \ldots, x_\alpha^{(i_\alpha')}, \ldots, x_p^{(i_p')})$ are adjacent if $\sum_{\alpha=1}^{p} |i_\alpha - i_\alpha'| = 1$. The internal net region $\omega_h^{(1)} = \{x_i \in G\}$ consists of the nodes, all of whose adjacent nodes belong to $\bar{G}$. The node x_i is a boundary node if at least one of its adjacent nodes does not belong to $\bar{G}$. Through the point $x_i \in \omega_h^{(1)}$ we draw the straight line $\mathcal{L}_\alpha$, parallel to the coordinate axis Ox_α and call the set of nodes of the net $\omega_h^{(1)}$ lying on $\mathcal{L}_\alpha$, including the boundary nodes $x^{(-\alpha)}$ and $x^{(+\alpha)}$ the chain $Ц_\alpha$ (the coordinate x_α increases on passing from $x^{(-\alpha)}$ to $x^{(+\alpha)}$). Let γ_α^- denote the set of left-hand boundary nodes $x^{(-\alpha)}$ and γ_α^+, the set of right-hand boundary nodes $x^{(+\alpha)}$ of all chains of a given directions $\gamma_\alpha = \gamma_\alpha^- + \gamma_\alpha^+$, $\gamma = \gamma_1 + \ldots + \gamma_p$ is the boundary of the region $\omega_h^{(1)}$, $\bar{\omega}_h^{(1)} = \omega_h^{(1)} + \gamma$.

2. The net $\omega_h^{(2)} = \{x_i \in G\}$. Every node $x_i \in G$ is an internal node. The boundary γ of the net $\omega_h^{(2)}$ consists only of points of the boundary Γ of the region G so that $\gamma \subset \Gamma$ (cf. [12]).

For regions G formed from parallelepipeds with their boundaries parallel to the coordinate planes (cf. [9]) the nets $\omega_h^{(1)}$ and $\omega_h^{(2)}$ coincide.

Both nets $\omega_h^{(1)}$ and $\omega_h^{(2)}$ are non-uniform, and the step of the net $h_\alpha^{(i_\alpha)} = x_\alpha^{(i_\alpha)} - x_\alpha^{(i_\alpha - 1)}$ for each direction α is a function of the coordinate x_α (number i_α). The net $\omega_h^{(2)}$ is non-uniform even when the net $\omega_h^{(1)}$ is uniform, but the region G is arbitrary, since near the boundary Γ the uniformity of the net is, generally speaking, destroyed.

The net can also be taken to be non-uniform with respect to the variable $t \in [0, T]$. Let $t_0 = 0, t_1, \ldots, t_j, \ldots, t_k = T$ be an arbitrary division of the segment $0 \leqslant t \leqslant T$ into K parts. Following [10] we can divide each of the segments $[t_j, t_{j+1}]$ into p (the number of dimensions) equal parts and introduce the intermediate (fractional) times

$$t_{j+\alpha/p} = t_j + \alpha\tau_{j+1}/p, \quad \alpha = 1, 2, \ldots, p-1,$$

where $\tau_{j+1} = t_{j+1} - t_j$ is the step of the basic net. The net $\omega_\tau = \{t_{j*} \in (0, T]\}$ contains both integral ($t = t_j$, $j^* = j$) and fractional times ($j^* = j + \alpha/p$): $j^* = 0, 1/p, \ldots, (p-1)/p, 1, \ldots, j, j + 1/p, \ldots, j + (p-1)/p, j+1, \ldots, K$. The set of nodes (x_i, t_{j*}), where $x_i \in \omega_h^{(k)}$, $t_{j*} \in \omega_\tau$, forms the space-time net $\Omega^{(k)} = \omega_h^{(k)} \times \omega_\tau$, $k = 1, 2$. The boundary S of the net $\Omega^{(k)}$ is defined in the same way as in [10],

$\bar{\Omega} = \Omega + S$.

As in [10] and [14] we shall use the following notation for net functions given on $\bar{\Omega}$.

$$y = y(x, t) = y(x_i, t_{j+\alpha/p}) = y^{j+\alpha/p}, \qquad h_\alpha = h_\alpha^{(i_\alpha)}, \qquad h_{\alpha+} = h_\alpha^{(i_\alpha+1)},$$
$$\hbar_\alpha = 0.5\,(h_\alpha + h_{\alpha+}),$$
$$x^{(\pm m_\alpha)} = x_i^{(\pm m_\alpha)} = (x_1^{(i_1)}, \ldots, x_{\alpha-1}^{(i_{\alpha-1})}, x_\alpha^{(i_\alpha \pm m)}, x_{\alpha+1}^{(i_{\alpha+1})}, \ldots, x_p^{(i_p)}), \quad m = 0.5, 1, 2,$$
$$x_\alpha^{(i_\alpha \pm 0.5)} = 0.5\,(x_\alpha^{(i_\alpha)} + x_\alpha^{(i_\alpha \pm 1)}), \qquad y^{(\pm m_\alpha)} = y\,(x^{(\pm m_\alpha)}, t),$$
$$y_{\bar{x}_\alpha} = (y - y^{(-1_\alpha)})/h_\alpha, \qquad y_{x_\alpha} = (y^{(+1_\alpha)} - y)/h_{\alpha+},$$
$$y_{\hat{x}_\alpha} = (y^{(+1_\alpha)} - y)/\hbar_\alpha = \frac{h_{\alpha+}}{\hbar_\alpha} y_{x_\alpha},$$
$$y_{\check{x}_\alpha} = (y - y^{(-1_\alpha)})/\hbar_\alpha = \frac{h_\alpha}{\hbar_\alpha} y_{\bar{x}_\alpha}, \qquad y_{\overset{\circ}{x}_\alpha} = \frac{1}{2}(y_{\hat{x}_\alpha} + y_{\check{x}_\alpha}) = (y^{(+1_\alpha)} - y^{(-1_\alpha)})/2\hbar_\alpha,$$
$$y_{\bar{t}} = (y^{j+1} - y^j)/\tau, \qquad y_{\bar{t}_\alpha} = (y^{j+\alpha/p} - y^{j+(\alpha-1)/p})/\tau, \qquad \tau = \tau_{j+1}.$$

We shall introduce other notation in the course of the argument.

If the step of the net ω_τ satisfies the condition

$$\tau_{\bar{t}} = O(\tau) \quad \text{or} \quad \tau = \check{\tau}\,(1 + O(\tau)) \qquad (\check{\tau} = \tau_j), \tag{3}$$

then $\omega_\tau = \omega_\tau^{**}$ is called a quasi-uniform net. If

$$\tau \leqslant m^* \check{\tau}, \tag{4}$$

where $m^* > 0$ is an arbitrary constant, then $\omega_\tau = \omega_\tau^*$ (and, correspondingly, $\Omega^{(k)} = \Omega_*^{(k)}$, $k = 1, 2$) is called a normal net.

3. HOMOGENEOUS SCHEMES ON NON-UNIFORM NETS ω_h

Before going on to construct local one-dimensional schemes for the problems I_k, $k = 0, 1, 2, 3$ let us first introduce a family of homogeneous difference schemes corresponding to the operator L_α, assuming the net ω_h to be non-uniform.

Let us consider an operator of the form

$$L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x)\,\frac{\partial y}{\partial x_\alpha}\right) + f(x), \; x = (x_1, \ldots, x_p).$$

We take the corresponding homogeneous difference scheme in the form

$$\Lambda_\alpha y = (a_\alpha(x)\, y_{\bar{x}_\alpha})_{\hat{x}_\alpha} + \varphi(x). \tag{5}$$

This scheme is defined if we can show a method for calculating the coefficients a_α and φ in terms of the functions $k_\alpha(x)$ and $f(x)$ which is suitable for an arbitrary non-uniform net.

We shall start from the method shown in [14] for $p = 1$, bearing in mind that k_α and f can have discontinuities of the first kind on hyperplanes parallel to the coordinate hyperplanes and passing through the nodes of the net.

Let us introduce the mean value $\tilde{f}(x)$ of the function $f(x)$ at the point $x \in \omega_h$ putting

$$\tilde{f}(x) = \frac{1}{2^p} \sum_{\sigma_1,\ldots,\sigma_p=1}^{2} f(x_{1,\sigma_1}, \ldots, x_{p,\sigma_p})\, h_1^{(\sigma_1)} \ldots h_p^{(\sigma_p)}/H, \quad H = \prod_{\alpha=1}^{p} \hbar_\alpha, \quad (6)$$

where $x_{\alpha,1} = x_\alpha - 0$, $x_{\alpha,2} = x_\alpha + 0$, $h_\alpha^{(1)} = h_\alpha$, $h_\alpha^{(2)} = h_{\alpha+}$, and each of the $\sigma_1, \ldots, \sigma_\alpha, \ldots, \sigma_p$ takes one of the values 1 or 2. In particular, when $p = 1$ and $p = 2$

$$\tilde{f}(x_1) = \frac{1}{2\hbar_1}(h_1 f(x_1 - 0) + h_{1+} f(x_1 + 0)),$$
$$\tilde{f}(x_1, x_2) = \frac{1}{4\hbar_1\hbar_2}[f(x_1 - 0, x_2 - 0)\, h_1 h_2 + f(x_1 - 0, x_2 + 0)\, h_1 h_{2+} + f(x_1 + 0, x_2 - 0)\, h_{1+} h_2 + f(x_1 + 0, x_2 + 0)\, h_{1+} h_{2+}].$$

If x is a point of continuity of the function f, then $\tilde{f}(x) = f(x)$. On a uniform net $(h_\alpha = h_{\alpha+})$ we have

$$\tilde{f}(x) = \frac{1}{2^p} \sum_{\sigma_1,\ldots,\sigma_p=1}^{2} f(x_{1,\sigma_1}, \ldots, x_{p,\sigma_p}). \quad (7)$$

By analogy with [14], let us consider the functions

$$\eta_0^{(1)}(s) = \begin{cases} 1, & s < 0 \\ 0, & s \geqslant 0 \end{cases}, \quad \eta_0^{(2)}(s) = \begin{cases} 0, & s \leqslant 0 \\ 1, & s > 0 \end{cases}, \quad \pi_0(s) = \begin{cases} 1, & s = 0 \\ 0, & s \neq 0 \end{cases}. \quad (8)$$

Let $F_p[\mu(s_1, \ldots, s_p)] = F_p[\mu(s)]$ be a linear non-negative functional which is defined in the parallelepiped $\{-0.5 \leqslant s_\alpha \leqslant 0.5,\ \alpha = 1, \ldots, p\}$ and satisfies the conditions

$$F_p[1] = 1, \quad F_p\Big[\prod_{\alpha=1}^{p} \eta_0^{(\sigma_\alpha)}(s_\alpha)\Big] = \frac{1}{2^p}, \quad F_p\Big[s_\alpha \prod_{k\neq\alpha}^{1-p} \eta_0^{(\sigma_k)}(s_k)\Big] = 0,$$
$$\alpha = 1, 2, \ldots, p. \quad (9)$$

Let us consider the function

$$f^{*}(s)=\frac{1}{H}\sum_{\sigma_1,\ldots,\sigma_p}^{1-2} f\left(x_1+s_1h_1^{(\sigma_1)},\ldots,x_p+s_ph_p^{(\sigma_p)}\right)h_1^{(\sigma_1)}\eta_0^{(\sigma_1)}(s_1)\ldots h_p^{(\sigma_p)}\eta_0^{(\sigma_p)}(s_p) \tag{10}$$

and define $\varphi(x)$ with the help of the formula

$$\varphi(x)=F_p\left[f^{*}(s)\right]=F_p\left[f^{*}(s_1,\ldots,s_p)\right]. \tag{11}$$

In the case of the simplest functional

$$F\left[f^{*}(s)\right]=\tilde{f}(x) \tag{12}$$

we obtain $\varphi(x)=\tilde{f}(x)$. On a uniform net, from conditions (9) it follows from (10) and (11) that

$$\varphi(x)=F_p\left[f(x_1+s_1h_1,\ldots,x_p+s_ph_p)\right]. \tag{13}$$

For the coefficient $a_\alpha(x)$ we shall use the formula

$$a_\alpha(x)=A\left[\tilde{k}_\alpha^{(\alpha)}(x+s_\alpha h_\alpha)\right], \tag{14}$$

where $A[\mu(s_\alpha)]$ is the one-dimensional functional of [14] and

$$\tilde{k}_\alpha^{(\alpha)}(x+s_\alpha h_\alpha)= \tag{15}$$

$$=\frac{1}{2^{p-1}}\sum_{\substack{\sigma_1,\ldots,\sigma_k,\ldots,\sigma_p\\(k\neq\alpha)}}^{1-2} k_\alpha(x_{1,\sigma_1},\ldots,x_{\alpha-1,\sigma_{\alpha-1}},x_\alpha+s_\alpha h_\alpha,x_{\alpha+1,\sigma_{\alpha+1}},\ldots,x_{p,\sigma_p})\,H_{\alpha,\sigma}/H_\alpha,$$

where

$$H_\alpha=\prod_{k\neq\alpha}^{1-p}\hbar_k,\quad H_{\alpha,\sigma}=\prod_{k\neq\alpha}^{1-p}h_k^{(\sigma_k)}.$$

Thus when calculating a_α we first carry out an averaging in accordance with (15) with respect to the variables $x_1,\ldots,x_{\alpha-1},x_{\alpha+1},\ldots,x_p$ and then operate with the pattern functional $A[\mu(s_\alpha)]$. The mean value (15) could be calculated by analogy with φ using the linear functional F_{p-1} of dimension $p-1$. In the case of two variables ($p=2$) we have

$$\tilde{k}_1^{(1)}(x+s_1h_1)=\frac{1}{2\hbar_2}\left(k_1(x_1+s_1h_1,\ x_2-0)\,h_2+k_1(x_1+s_1h_1,\ x_2+0)\,h_{2+}\right).$$

It is not difficult to see that the approximation error $\varphi-\tilde{f}$ can be put in the form

$$\varphi(x)-\tilde{f}(x)=\sum_{\alpha=1}^{p}(\mu_\alpha)_{\hat{x}_\alpha}+\cdots,\qquad \mu_\alpha=O(h_\alpha^2). \tag{16}$$

The dots here and everywhere below denote terms of the second order with respect to $\hbar_1, \ldots, \hbar_p$.

In fact, expanding $f^*(s)$, defined by formula (10), in the neighbourhood of the point $x^{(\sigma)} = (x_{1,\sigma_1}, \ldots, x_{p,\sigma_p})$:

$$f(x_1 + s_1 h_1^{(\sigma_1)}, \ldots, x_p + s_p h_p^{(\sigma_p)}) = f(x^{(\sigma)}) + \sum_{\alpha=1}^{p} \frac{\partial f}{\partial x_\alpha}(x^{(\sigma)})\, s_\alpha h_\alpha^{(\sigma_\alpha)} + \ldots.$$

Putting this expression in (10) and (11) we obtain

$$\varphi(x) = \tilde{f}(x) + \sum_{\alpha=1}^{p} \sum_{\sigma_\beta,\, \beta\neq\alpha}^{1-2} \left[h_{\alpha+}^2 \frac{\partial f}{\partial x_\alpha}(x^{(\sigma)}) \Big|_{\sigma_\alpha=2} - h_\alpha^2 \frac{\partial f}{\partial x_\alpha}(x^{(\sigma)}) \Big|_{\sigma_\alpha=1} \right] \times$$

$$\times F_p\left[s_\alpha \eta_0^{(2)}(s_\alpha) \prod_{\beta\neq\alpha}^{1-p} \eta_0^{(\sigma_\beta)}(s_\beta)\right] \prod_{\beta\neq\alpha}^{1-p} h_\beta^{(\sigma_\beta)} / H + \cdots.$$

Noting that

$$H = H_\alpha \times \hbar_\alpha, \qquad H_\alpha = \prod_{\beta\neq\alpha}^{1-p} \hbar_\beta, \qquad \prod_{\beta\neq\alpha}^{1-p} h_\beta^{(\sigma_p)} \times H_\alpha^{-1} = 0(1) \text{ and}$$

$$h_{\alpha+}^2 \frac{\partial f}{\partial x_\alpha}(x^{(\sigma)}) \Big|_{\sigma_\alpha=2} - h_\alpha^2 \frac{\partial f}{\partial x_\alpha}(x^{(\sigma)}) \Big|_{\sigma_\alpha=1} = \left(h_\alpha^2 \frac{\partial f}{\partial x_\alpha}(x^{(\sigma)}) \Big|_{\sigma_\alpha=1} \right)_{\hat{x}_\alpha} + \cdots,$$

and summing over $\sigma_\beta = 1, 2$, $\beta = 1, 2, \ldots, p$, $\beta \neq \alpha$, we arrive at (16).

By analogy with [14] we find

$$(a_\alpha(x)\, u_{\bar{x}_\alpha})_{\hat{x}_\alpha} - \widetilde{\frac{\partial}{\partial x_\alpha}\left(k_\alpha(x) \frac{\partial u}{\partial x_\alpha}\right)} = (\mu_\alpha)_{\hat{x}_\alpha} + \cdots, \qquad \mu_\alpha = O(h_\alpha^2), \tag{17}$$

where the wavy line denotes averaging according to formula (6).

Note. In the case of the simplest functional (12) $\varphi(x) - \tilde{f}(x) = 0$.

4. LOCAL ONE-DIMENSIONAL SCHEMES ON NON-UNIFORM NETS

The basic idea of the method for writing down local one-dimensional schemes for the equation (1) consists in using homogeneous difference schemes corresponding to the one-dimensional parabolic equation

$$\mathscr{P}_\alpha u = L_\alpha u - \frac{1}{p} c \frac{\partial u}{\partial t} + f_\alpha = 0, \qquad \sum_{\alpha=1}^{p} f_\alpha = f \tag{18}$$

at each moment of time $t_j \in \omega_\tau$.

Let $\Lambda_\alpha y$ denote the one-dimensional difference scheme corresponding to the operator $L_\alpha u$. Then we can write the homogeneous scheme Π_α for (18) in the form

$$\Pi_\alpha y = \Lambda_\alpha y - \rho y_{\bar{t}_\alpha} + \varphi_\alpha = 0, \qquad \Lambda_\alpha y = \Lambda_\alpha y^{j+\alpha/p}. \tag{19}$$

We shall assume that ρ does not depend on α, i.e. for all $\alpha = 1, 2, \ldots, p$ the coefficient ρ is calculated according to the same formula:

$$\rho = \rho(x, t_{j+1}) = F_p\,[c^*(s, t_{j+1})], \tag{20}$$

where $c^*(s, t)$ is defined by the formula (10) and $F_p[\mu(s)] = F_p[\mu(s_1, \ldots, s_p)]$ is the linear non-decreasing functional introduced in Para. 3.

The coefficients which enter into Λ_α as well as φ_α will be calculated with the help of the one-dimensional functionals $A[\mu(s)]$ and the linear functional $F_1[\mu(s)]$ $(-0.5 \leqslant s \leqslant 0.5)$ used in [14] ($F_1[1] = 1$, $F_1[s] = 0$, $F_1[\eta_0^{(\sigma)}(s)] = \frac{1}{2}$, $F_1[\pi_0(s)] = 0$, $\sigma = 1, 2$). These functionals operate on the corresponding functions before they are averaged with respect to the variable x_β, $\beta \neq \alpha$, $\beta = 1, \ldots, p$ according to (15). We shall write these mean values $\tilde{k}_\alpha(x + s_\alpha h_\alpha, t, u)$, $\tilde{r}_\alpha(x + s_\alpha h_\alpha, t, u)$, $\tilde{f}_\alpha(x + s_\alpha h_\alpha, t, u)$, omitting the upper index (α); then

$$\left.\begin{aligned} a_\alpha(x, t, u) &= A\,[\tilde{k}_\alpha(x + s_\alpha h_\alpha, t, u)], \\ b_\alpha^-(x, t, u) &= F_1\,[\tilde{r}_\alpha(x + s_\alpha h_\alpha, t, u)\,\eta_0^{(1)}(s_\alpha)], \\ b_\alpha^+(x, t, u) &= F_1\,[\tilde{r}_\alpha(x + s_\alpha h_\alpha, t, u)\,\eta_0^{(2)}(s_\alpha)], \\ \varphi_\alpha(x, t, u) &= [\varphi_\alpha^- h_\alpha + \varphi_\alpha^+ h_{\alpha+}]/\hbar_\alpha, \end{aligned}\right\} \tag{21}$$

where $\varphi_\alpha^\pm(x, t, u)$ are expressed according to the same formulae as $b_\alpha^\pm(x, t, u)$.

Let us first consider the problem I_1. In this case

$$\begin{gathered} \Lambda_\alpha y = (a_\alpha(x, t)\, y_{\bar{x}_\alpha})_{\hat{x}_\alpha} + b_\alpha^+(x, t, \check{y})\, y_{\hat{x}_\alpha} + b_\alpha^-(x, t, \check{y})\, y_{\check{x}_\alpha}, \\ \varphi_\alpha = \varphi_\alpha(x, t, \check{y}), \end{gathered} \tag{22_1}$$

where $y = y^{j+\alpha/p}$, $\check{y} = y^{j+(\alpha-1)/p}$. In [10] we showed how the coefficients entering in (22_1) could be chosen at any moment of time $t \in [t_j, t_{j+1}]$. We obtained schemes which were equivalent with respect to their order of accuracy. To simplify the printing we shall put $t = t_{j+1}$ in (22_1) below. It turns out that, in general, the smoothness requirements with respect to t for the coefficients of the differential equation are weakened.

The schemes Π_α for the different equations 1_k, $k = 1, 2, 3$ differ only in the expressions for $\Lambda_\alpha y$ and φ_α. We shall therefore write down these expressions for the equations 1_2 and 1_3. For the quasi-linear equation (1_2) we shall use the homogeneous scheme

$$\Lambda_\alpha y = (a_\alpha(x, t, (y^*)^{(\beta)}) y_{\bar{x}_\alpha})_{\hat{x}_\alpha} + b_\alpha^+(x, t, \check{y})\, y_{\hat{x}_\alpha} + b_\alpha^-(x, t, \check{y})\, y_{\check{x}_\alpha}, \tag{22_2}$$

$$\varphi_\alpha = \varphi_\alpha(x, t, \check{y}),$$

where $(y^*)^{(\beta)} = \beta y^* + (1-\beta)\check{y}^*,\ 0 \leqslant \beta \leqslant 1,\ y^* = 0.5\,(y + y^{(-1_\alpha)}),\ t = t_{j+1}$. Besides schemes (22_1), (22_2) the following schemes are of theoretical and practical interest:

$$\Lambda_\alpha y = (a_\alpha(x, t)\, y_{\bar{x}_\alpha})_{\hat{x}_\alpha} + b_{\alpha-1}^+(x, t, \check{y})\, \check{y}_{\hat{x}_{\alpha-1}} + b_{\alpha-1}^-(x, t, \check{y})\, \check{y}_{\check{x}_{\alpha-1}}, \tag{$22_{1'}$}$$

$$\varphi_\alpha = \varphi_\alpha(x, t, \check{y}),$$

$$\Lambda_\alpha y = (a_\alpha(x, t, \check{y}^*)\, y_{\bar{x}_\alpha})_{\hat{x}_\alpha} + b_{\alpha-1}^+(x, t, \check{y})\, \check{y}_{\hat{x}_{\alpha-1}} + b_{\alpha-1}^-(x, t, \check{y})\, \check{y}_{\check{x}_{\alpha-1}}, \tag{$22_{2'}$}$$

$$\varphi_\alpha = \varphi_\alpha(x, t, \check{y}),$$

where $b_{\alpha-1}^{\pm} = b_p^{\pm}(x, t_j, y^j)$, $\check{y}_{x_{\alpha-1}} = y^j_{x_p}$ for $\alpha = 1$. These schemes correspond to different methods of partitioning the operator $L = \sum_{\alpha=1}^{p} L_\alpha$:

$$Lu = \sum_{\alpha=1}^{p} L'_\alpha u, \qquad L'_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x, t, u)\frac{\partial u}{\partial x_\alpha}\right) + r_{\alpha-1}(x, t, u)\frac{\partial u}{\partial x_{\alpha-1}}$$

$$\text{for } \alpha > 1,$$

$$L'_1 u = \frac{\partial}{\partial x_1}\left(k_1(x, t, u)\frac{\partial u}{\partial x_1}\right) + r_p(x, t, u)\frac{\partial u}{\partial x_p} \quad \text{for } \alpha = 1.$$

The coefficients a_α and ρ satisfy the conditions

$$a_\alpha \geqslant c_1 > 0, \qquad \rho \geqslant c_2 > 0, \tag{23}$$

and r_α ($b_\alpha^{\pm}$ too) can have any sign. Therefore in solving equations (22_1), (22_2) for example, with respect to y by the method of successive substitution [21] it is, generally speaking, required that the step $h_\alpha \leqslant h_0$ shall be sufficiently small, where h_0 depends on $\max(|b_\alpha^{\pm}|/a_\alpha)$ (cf. [16]). The difference schemes $(22_{1'})$, $(22_{2'})$ do not possess this defect.

We shall not give the scheme for (1_0) since it follows from (22_1) in the case $r_\alpha = r_\alpha(x, t)$, $f = f(x, t) - qu$. We have given equation (1_0) in Para. 1 in order to establish the connection with the work [10].

Let us now formulate the boundary conditions. On the net $\omega_h^{(1)}$, by analogy with [10], the boundary conditions are set up by means of an extrapolation $u|_\Gamma$ on γ_α with the help of linear interpolation, which gives for $y = y^{j+\alpha/p}$

$$y = \beta_\alpha^- y^{(+1_\alpha)} + (1-\beta_\alpha^-)\, u_1(x_1^{(-\alpha)}, t_{j+1}), \qquad x \in \gamma_\alpha^-;$$
$$y = \beta_\alpha^+ y^{(+1_\alpha)} + (1-\beta_\alpha^+)\, u_1(x_r^{(+\alpha)}, t_{j+1}), \qquad x \in \gamma_\alpha^+, \tag{24}$$

where $x_1^{(-\alpha)}$ is the point of the boundary Γ nearest to $x^{(-\alpha)} \in \gamma_\alpha^-$ and $x_r^{(+\alpha)} \in \Gamma$ is the nearest point to $x^{(+\alpha)} \in \gamma_\alpha^+$ (the points $x_1^{(-\alpha)}$, $x_r^{(+\alpha)}$ and $x^{(\pm\alpha)}$ lie on the straight line $\mathscr{L}_\alpha$), $\beta_\alpha^\pm = \varkappa_\alpha^\pm/(1+\varkappa_\alpha^\pm)$, $\varkappa_\alpha^- h_{\alpha+}$ is the distance between $x_1^{(-\alpha)}$ and $x^{(-\alpha)}$, $\kappa_\alpha^+ h_\alpha$ is the distance between $x_r^{(+\alpha)}$ and $x^{(+\alpha)}$ and $0 \leqslant \beta_\alpha^\pm < 1$. We shall only consider nets $\omega_h^{(1)}$ for which $\beta_\alpha^\pm \leqslant \beta^* < 1$ where β^* is a constant.* In other respects $\omega_h^{(1)}$ is an arbitrary non-uniform net. On the net $\omega_h^{(2)}$ the boundary conditions have the form

$$y^{j+\alpha/p} = u_1(x^{(\pm\alpha)}, t_{j+1}) \quad \text{for } x = x^{(\pm\alpha)} \in \gamma_\alpha^\pm, \qquad \alpha = 1, \ldots, p, \tag{25}$$

which formally corresponds to the case $\beta_\alpha^\pm = 0$. Therefore we shall always write the boundary conditions in the form (24) below and, on passing to the net $\omega_h^{(2)}$, we shall put $\beta_\alpha^\pm = 0$.

It is clear from (24) and (25) that the boundary values $u_1 - u|_\Gamma$ for all $\alpha = 1, 2, \ldots, p$ are taken at time $t = t_{j+1}$. This does not restrict the generality of our argument, since if we take $u|_\Gamma$ at any time $t \in [t_j, t_{j+1}]$ we obtain schemes which are equivalent with respect to their order of accuracy.

Thus, we set the following difference problem I_k in correspondence to the problem I_k $(k = 1, 2)$:

$$\left.\begin{aligned} &\Lambda_\alpha y - \rho y_{\bar{t}_\alpha} + \varphi_\alpha = 0, \qquad \alpha = 1, 2, \ldots, p, \quad (x, t) \in \Omega, \\ &y = \beta_\alpha^\pm y^{(\mp 1_\alpha)} + (1-\beta_\alpha^\pm)\, u_1^{j+1} \quad \text{for } x \in \gamma_\alpha^\pm, \\ &y(x, 0) = u_0(x) \qquad \text{for } x \in \bar{\omega}_h, \end{aligned}\right\} \tag{II}$$

where Λ_α is defined by formulae (22_1) and (22_2). Depending on the form of Λ_α we obtain the problems (schemes) II_1 and II_2 corresponding to problems I_1 and I_2. If Λ_α, φ_α are defined by formulae $(22_1{'})$ or $(22_2{'})$ we obtain the scheme $\mathrm{II}_1{'}$ or $\mathrm{II}_2{'}$.

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* We shall always assume below that this conditions is satisfied.

We set the problem I_3 in correspondence with the scheme II_3 for which

$$\left.\begin{aligned} &\Lambda_\alpha y = (a_\alpha (x, t, \check{y}^*)\, y_{\bar{x}_\alpha})_{\hat{x}_\alpha}, \qquad \alpha = 1, 2, \ldots, p, \\ &\varphi_\alpha = \delta_{\alpha, p}\, \varphi (x,\ t,\ \check{y}, y^{j+1/p}_{\overset{\circ}{x}_1}, \ldots, y^{j+1}_{\overset{\circ}{x}_p}), \\ &\delta_{\alpha, p} = \begin{cases} 1, & \alpha = p \\ 0, & \alpha \neq p \end{cases}, \qquad t = t_{j+1}. \end{aligned}\right\} \tag{22$_3$}$$

We shall consider the schemes II_1', II_2' and II_3 only on arbitrary non-uniform nets $\omega_h^{(1)}$ and on normal nets ω_τ^*.

The local one-dimensional scheme II is obviously homogeneous (for each α) with respect to space and cyclically homogeneous with respect to time (with period p).

Note. The decomposition of f into the sum $f = \sum\limits_{\alpha=1}^{p} f_\alpha$ is done for convenience in the computing. We could put, for example $f_\alpha = 0$, $\alpha = 1, 2, \ldots, p - 1$, $f_p = f$ and, correspondingly, $\varphi_\alpha = 0$, $\alpha = 1, \ldots, p - 1$, $\varphi_p = \varphi$.

5. THE APPROXIMATION ERROR

Let $u = u(x, t)$ be the solution of problem I and $y = y(x, t)$ the solution of the difference problem II. Let us examine their difference, which characterises the accuracy of the scheme II, putting $z^{j+\alpha/p} = y^{j+\alpha/p} - u^{j+1}$ for $\alpha = 1, 2, \ldots, p$, $z^j = y^i - u^j$. For the net function $z = z(x, t)$ we obtain the following conditions:

$$\left.\begin{aligned} \rho z_{\bar{t}_\alpha} &= (a_\alpha z_{\bar{x}_\alpha})_{\hat{x}_\alpha} + Q_\alpha (z) + \Psi_\alpha, \\ z &= \beta_\alpha^\pm (z^{(\mp 1_\alpha)} + \nu_\alpha^\pm) \quad \text{for } x \in \gamma_\alpha^\pm, \\ z (x, 0) &= 0 \quad \text{for } x \in \bar{\omega}_h, \end{aligned}\right\} \tag{III}$$

where $a_\alpha = a_\alpha(x, t_{j+1})$ for the scheme II_1, $a_\alpha = a_\alpha (x, t_{j+1}, (y^*)^{(\beta)})$ for the scheme II_2 and $Q_\alpha(z)$ is an expression containing the earliest terms (z, $z_{\hat{x}_\alpha}$, $z_{\check{x}_\alpha}$ and so on) and is a special case of the expression

$$Q_\alpha (z) = (g_{\alpha 1}^+ z)_{\hat{x}_\alpha} + (g_{\alpha 1}^- z)_{\check{x}_\alpha} + (g_{\alpha 2}^+ \check{z})_{\hat{x}_\alpha} + (g_{\alpha 2}^- \check{z})_{\check{x}_\alpha} + b_\alpha^+ \hat{z}_{\hat{x}_\alpha} + \\ + b_\alpha^- z_{\check{x}_\alpha} + d_{\alpha 1} z + d_{\alpha 2} \check{z}. \tag{26}$$

We shall not write out the expressions for the coefficients. We note only that in the case of the scheme II_1 we must put $g^{\pm}_{\alpha m} = 0$, $d_{\alpha 1} = 0$, $m = 1, 2$, $\alpha = 1, 2, \ldots, p$. Due to the assumptions made in Para. 1, 3 and 4

$$a_\alpha \geqslant c_1 > 0, \quad \rho \geqslant c_2 > 0, \quad |d_{\alpha m}| \leqslant c_3, \quad |b_\alpha^{\pm}| \leqslant c_4, \quad |g^{\pm}_{\alpha m}| \leqslant c_5, \qquad m = 1, 2,$$

where $c_1, \ldots, c_5$ are positive constants which do not depend on the net.

Let us show that $g^{\pm}_{\alpha 1} = g^{\pm}_{\alpha 1}(x, t_{j+1}, y^*)$, $b^{\pm}_\alpha = b^{\pm}_\alpha(x, t_{i+1}, \check{y})$ depend on y and so cannot be differentiated with respect to x or to t. For the local approximation error of the scheme II_2 and, therefore, of the scheme II_1, we obtain

$$\Psi_\alpha = \Psi_\alpha^{(1)} - \rho(x, t_{j+1})\, u_{\bar{t}} \times \delta_{\alpha,1} + \varphi_\alpha, \qquad \delta_{\alpha,1} = \begin{cases} 1, & \alpha = 1, \\ 0, & \alpha \neq 1, \end{cases} \tag{27}$$

$$\Psi_\alpha^{(1)} = \left(a_\alpha(x, t_{j+1}, (u^*)^{(\beta)})\, u^{j+1}_{\bar{x}_\alpha}\right)_{\hat{x}_\alpha} + b^+_\alpha(x, t_{j+1}, \bar{u})\, u^{j+1}_{\hat{x}_\alpha} +$$

$$+ b^-_\alpha(x, t_{j+1}, \bar{u})\, u^{j+1}_{\check{x}_\alpha}, \quad \varphi_\alpha = \varphi_\alpha(x, t_{j+1}, \bar{u}),$$

where $\bar{u} = u^{j+1}$ for $\alpha > 1$, $\bar{u} = u^j$ for $\alpha = 1$, $(u^*)^{(\beta)} = (u^*)^{j+1}$ for $\alpha > 1$, $(u^*)^{(\beta)} = \beta(u^*)^{j+1} + (1 - \beta)\ (u^*)^j$ for $\alpha = 1$.

It is clear from this that $\Psi_\alpha^{(1)}$ represents the error of approximation of the scheme $\Lambda_\alpha y$. It follows from Para. 3 and 4 and from [14] that

$$\Psi_\alpha^{(1)} = (\widetilde{L_\alpha u})^{j+1} + (\mu_\alpha^{(1)})_{\hat{x}_\alpha} + \delta_{\alpha,1} O(\tau) + \cdots, \qquad \mu_\alpha^{(1)} = O(h_\alpha^2).$$

Noting also that

$$\varphi_\alpha = \widetilde{f_\alpha}(x, t_{j+1}, u^{j+1}) + (\mu_\alpha^{(2)})_{\hat{x}_\alpha} + \delta_{\alpha,1} O(\tau) + \cdots, \qquad \mu_\alpha^{(2)} = O(h_\alpha^2),$$

$$\rho = \tilde{c}(x, t_{j+1}) + \sum_{\alpha=1}^{p} (\mu_\alpha^{(3)})_{\hat{x}_\alpha} + \cdots, \qquad \mu_\alpha^{(3)} = O(h_\alpha^2),$$

we see that

$$\left.\begin{aligned} &\Psi_\alpha = \mathring{\psi}_\alpha + \psi_\alpha, \qquad \mathring{\psi}_\alpha = \mathscr{F}^*_\alpha u = \left(\widetilde{L_\alpha} u - \delta_{\alpha,1}\, c\, \widetilde{\frac{\partial u}{\partial t}} + \widetilde{f_\alpha}\right)^{j+1}, \\ &\psi_\alpha = (\mu_\alpha)_{\hat{x}_\alpha} + \psi^*_\alpha + \delta_{\alpha,1} \sum_{k=1}^{p} [(\mu^*_k)_{\hat{x}_k} + \psi^{**}_k], \quad \delta_{\alpha,1} = \begin{cases} 1, \ \alpha = 1, \\ 0, \ \alpha \neq 1, \end{cases} \\ &\mu_\alpha = O(h_\alpha^2), \quad \mu^*_\alpha = O(h_\alpha^2), \quad \psi^{**}_\alpha = O(\hbar_\alpha^2) + O(\tau), \quad \psi^*_\alpha = O(\hbar_\alpha^2). \end{aligned}\right\} \tag{28}$$

It is clear from this that the principal part $\mathring{\psi}_\alpha$ of the approximation error satisfies the condition

$$\sum_{\alpha=1}^{p} \mathring{\psi}_\alpha = 0. \tag{29}$$

We assume here that we have satisfied conditions A, under which on a uniform net in the class of continuous coefficients Λ_α, φ_α, ρ have the maximum order of approximation

$$\Psi_\alpha^{(1)} = L_\alpha u + O(h^2), \qquad \rho = c + O(h^2), \qquad \varphi_\alpha = f_\alpha + O(h_\alpha^2).$$

If the node $x \in \omega_h$ lies on the hyperplane of discontinuity of k_α, c and f perpendicular to Ox_α and conditions A are satisfied to the left and right of this hyperplane, then formulae (28) for Ψ_α still hold.

The approximation error $\Psi = \sum_{\alpha=1}^{p} \Psi_\alpha$ of the local one-dimensional scheme $\Pi = \{\Pi_\alpha\}$ has the form

$$\Psi = \sum_{\alpha=1}^{p} \Psi_\alpha = \sum_{\alpha=1}^{p} [(\bar{\mu}_\alpha)_{\hat{x}_\alpha} + \bar{\psi}_\alpha], \qquad \bar{\mu}_\alpha = \mu_\alpha + \mu_\alpha^* = O(h_\alpha^2),$$
$$\bar{\psi}_\alpha = O(\hbar_\alpha^2) + O(\tau). \tag{30}$$

In proving the convergence of a local one-dimensional scheme on a sequence of non-uniform nets we must overcome two difficulties: (1) the absence of an approximation at any time $t = t_{j+\alpha/p}$, $\mathring{\Psi}_\alpha = O(1)$, (2) the lowering of the order of local approximation with respect to space due to the non-uniformity of the nets ω_h.

For the error z of problem II_3 we also obtain problem III_3 where $Q_\alpha(z)$ is the expression defined by formula (3_3) of Section 2. The approximation error Ψ_α of the scheme II_3 can be represented by formulae (4) and (5) of Section 2 and we obtain

$$\mathring{\psi}_\alpha = \left[\widetilde{L_\alpha u} - \delta_{\alpha,1}\, \widetilde{c\frac{\partial u}{\partial t}} + \delta_{\alpha,p} \tilde{f}\left(x, t, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_p}\right)\right]^{j+1}, \qquad \sum_{\alpha=1}^{p} \mathring{\psi}_\alpha = 0,$$
$$\mu_{\alpha\beta} = \mu_\alpha \delta_{\alpha\beta} + \mu_\beta^* \delta_{\alpha_1} + \mu_\beta^{**}\delta_{\alpha p}, \qquad \mu_\alpha = O(h_\alpha^2),$$
$$\mu_\beta^* = O(h_\beta^2), \qquad \mu_\beta^{**} = O(h_\beta^2).$$

Formula (28) for ψ_α is also a special case of (2.5) for

$$\mu_{\alpha\beta} = \mu_\alpha \delta_{\alpha\beta} + \mu_\beta^* \delta_{\alpha 1}, \qquad \mu_\alpha = O(h_\alpha^2), \qquad \mu_\beta^* = O(h_\beta^2).$$

Calculation shows that (2.4) and (2.5) are also true for the problems

$\mathrm{III}_{1'}$, $\mathrm{III}_{2'}$ corresponding to the schemes $\mathrm{II}_{1'}$, $\mathrm{II}_{2'}$. The expressions for $Q_\alpha^{(1')}$ and $Q_\alpha^{(2')}$ are given in Section 2, Para. 1.

In fact, in Section 2 we consider problems III_k, $k = 1, 2, 3, 1', 2'$ which are more general than the problems which are obtained for the error $z = y - u$ of the scheme II_k. This enables us to use the *a priori* estimates of Section 2 to estimate the accuracy not only of the initial schemes, defined above, but also of a wider class of schemes.

2. *A priori* estimates

1. STATEMENT OF THE PROBLEM

When studying the question of the error of the schemes II_k, $k = 1, 2, 3, 1', 2'$ we obtained linear equations in Section 1 for the net function z in all cases, distinguished only by the earlier terms $Q_\alpha(z)$. It is therefore natural to formulate the general problem and to study the special cases corresponding to specific schemes. We shall consider the following problem for $z = z(x, t)$, given on $\overline{\Omega}$:

$$\rho z_{\bar{t}_\alpha} - \Lambda_\alpha^0 z + d_\alpha z = Q_\alpha(z) + \Psi_\alpha, \qquad \Lambda_\alpha^0 z = (a_\alpha z_{\bar{x}_\alpha})_{\hat{x}_\alpha}, \quad \alpha = 1, \ldots, p, \quad (1)$$

$$z = \beta_\alpha^\pm (z^{(\mp 1_\alpha)} + \nu_\alpha^\pm), \quad x \in \gamma_\alpha^\pm; \qquad z(x, 0) = 0, \quad x \in \overline{\omega}_h, \quad (2)$$

where $Q_\alpha(z)$ is one of the expressions:

$$Q_\alpha^{(1)}(z) = b_\alpha^+ z_{\hat{x}_\alpha} + b_\alpha^- z_{\check{x}_\alpha} + d_{\alpha 2}\check{z}, \quad (3_1)$$

$$Q_\alpha^{(2)}(z) = Q_\alpha^{(1)}(z) + Q_\alpha^*(z), \qquad Q_\alpha^*(z) = (g_{\alpha 1}^+ z)_{\hat{x}_\alpha} + (g_{\alpha 1}^- z)_{\check{x}_\alpha} + (g_{\alpha 2}^+ \check{z})_{\hat{x}_\alpha} + (g_{\alpha 2}^- \check{z})_{\check{x}_\alpha}, \quad (3_2)$$

$$\left.\begin{aligned} Q_\alpha^{(3)}(z) &= Q_\alpha^{**}(z) + d_{\alpha 2}\check{z} + \delta_{\alpha,p} \sum_{\alpha=1}^{p} (b_\alpha^+ z_{\hat{x}_\alpha}^{j+\alpha/p} + b_\alpha^- z_{\check{x}_\alpha}^{j+\alpha/p}), \\ Q_\alpha^{**}(z) &= (g_{\alpha 2}^+ \check{z})_{\hat{x}_\alpha} + (g_{\alpha 2}^- \check{z})_{\check{x}_\alpha}, \end{aligned}\right\} \quad (3_3)$$

$$\left.\begin{aligned} Q_\alpha^{(1')}(z) &= b_{\alpha-1}^+ \check{z}_{\hat{x}_{\alpha-1}} + b_{\alpha-1}^- \check{z}_{\check{x}_{\alpha-1}} + d_{\alpha 2}\check{z} \quad \text{for } \alpha > 1, \\ Q_1^{(1')}(z) &= (b_p^+ z_{\hat{x}_p} + b_p^- z_{\check{x}_p})^j + d_{p2} z^j, \end{aligned}\right. \quad (3_{1'})$$

$$Q_\alpha^{(2')}(z) = Q_\alpha^{(1')}(z) + Q_\alpha^{**}(z). \quad (3_{2'})$$

The net function Ψ_α is given by the expressions

$$\Psi_\alpha = \mathring{\psi}_\alpha + \psi_\alpha, \qquad \sum_{\alpha=1}^{p} \mathring{\psi}_\alpha = 0, \qquad \mathring{\psi}_\alpha = O(1), \tag{4}$$

$$\psi_\alpha = \sum_{\beta=1}^{p} [(\mu_{\alpha\beta})_{\hat{x}_\beta} + \psi_{\alpha\beta}], \qquad \mu_{\alpha\beta} = O(h_\beta^2), \qquad \psi^*_{\alpha\beta} = O(\hbar_\beta^2) + O(\tau). \tag{5}$$

We shall assume that the following conditions are satisfied:

$$a_\alpha \geqslant c_1 > 0, \qquad \rho \geqslant c_2 > 0, \qquad |d_{\alpha 2}| \leqslant c_3, \qquad |b^\pm_{\alpha k}| \leqslant c_4,$$

$$|g^\pm_{\alpha k}| \leqslant c_5, \qquad |\rho_{\bar{t}}| \leqslant c_6, \quad k = 1, 2, \tag{6}$$

$$d_\alpha \geqslant M^* \text{ for } \tau < \tau_0, \text{ if } g^\pm_{\alpha k} = 0, \tag{7}$$

$$d_\alpha \geqslant M^* 2^n \text{ for } \tau < \tau_0/2^n, \text{ if } g^\pm_{\alpha k} \neq 0 \quad (n = 1, 2, \ldots), \tag{7'}$$

where $c_1, \ldots, c_6$ are positive constants which do not depend on the net, M^* is an arbitrary positive constant and $\tau_0 > 0$ is a constant which depends on $c_1, \ldots, c_5$ and M^*. Let us explain conditions (7) and (7'). When deriving the energy identities which we have used to obtain the necessary *a priori* estimates we first make the transformation (cf. [14])

$$z = v \times w, \tag{8}$$

where w is a net function defined by the conditions

$$w_{\bar{t}_\alpha} = \overline{M}\check{w} \text{ for } w^{j+\alpha/p} = (1 + \overline{M}\tau)\, w^{j+(\alpha-1)/p}, \qquad w(x, 0) = 1, \tag{8'}$$

and $\overline{M}$ is an arbitrary positive constant which is chosen so that conditions (7) and (7') are satisfied, where M^* and τ_0 are given constants depending only on $c_1, \ldots, c_5$. If $g^\pm_{\alpha k} \neq 0$ we put $\overline{M} = \overline{M}_0 \times 2^n$, where $\overline{M}_0$ is an arbitrary positive constant and the whole number n is the rank of the energy identity.

We note at once that on an arbitrary non-uniform net ω_τ w has the estimate

$$1 < w^{j+\alpha/p} \leqslant e^{p\overline{M}t_{j+1}}, \tag{9}$$

since $w^{j+\alpha/p} \leqslant e^{\overline{M}\tau_{j+1}} w^{j+(\alpha-1)/p}$, $w^j \leqslant e^{\overline{M}p\tau_j} w^{j-1}$. To simplify our argument we shall assume that the transformation (8) has already been carried out and retain the former notation z for the unknown function (i.e. we replace v by z).

We shall call the problem defined by conditions (1), (2), (3_k), (4)-(7), where $k = 1, 2, 3, 1', 2'$ problem III_k or III_k^0 if $\nu_\alpha^\pm = 0$.

We shall direct our main attention to deriving *a priori* estimates for problems III_1 and III_2. For problem III_2 on the net $\Omega^{(1)}$ we shall assume everywhere that

$$g_{\alpha 2}^{\pm} = 0, \qquad \alpha = 1, 2, \ldots, p. \tag{10}$$

We note that scheme III_1 coincides with the scheme considered in [10] on the uniform net $\Omega^{(1)}$. For this scheme the maximum principle holds and we have the *a priori* estimate of the form

$$\| z(x, t) \|_0 \leqslant M (\| \nu(x, t) \|_{0,s} + \overline{\| \Psi(x, t) \|_0}), \quad t \in \omega_\tau,$$

$$\| \nu(x, t) \|_{0,s} = \max_{0 < t' \leqslant t,\ \alpha = 1, \ldots, p} \| \nu(x, t') \|_{0,\gamma_\alpha}, \qquad \| \nu(x, t) \|_{0,\gamma_\alpha} =$$

$$= \max_{x \in \gamma_\alpha = \gamma_\alpha^+ + \gamma_\alpha^-} | \nu_\alpha^\pm (x, t) |, \tag{11}$$

$$\overline{\| \Psi(x, t_{j+1}) \|_0} = \left(\sum_{j'=1}^{j+1} \tau_{j'} \| \Psi(x, t_{j'}) \|_0^2 \right)^{1/2}, \qquad \| \Psi \|_0 = \max_{x \in \omega_h,\ \alpha = 1, \ldots, p} | \Psi_\alpha(x, t) |.$$

The estimate of $\| z \|_0$ in terms of $\| \Psi \|_0$ is too crude for an estimate of the order of accuracy on a non-uniform net. Below we find estimates of $\| z \|_0$ in terms of the analogue of the one-dimensional norm $\| \Psi \|_3$ used in [14] and these enable us to show that the scheme has the same order of accuracy as in the case of a uniform net. The method of integral or energy inequalities explained below is a natural development of the method which we have used before (cf. [14]-[17]) for one-dimensional problems. Estimates "in the mean" are comparatively simple to obtain for all the problems III_k, $k = 1, 2, 3, 1', 2'$. However, since it is in practice very desirable to have uniform estimates of accuracy of numerical algorithms, we have given considerable attention to uniform estimates, first for problem III_1 corresponding to scheme II_1 for the quasi-linear equation (1.1_1). In [14] we used estimates in the mean successfully to obtain uniform estimates. In the multi-dimensional case, unfortunately, this method is inapplicable. Therefore the uniform estimates of the solution of problem III_2 are more crude than the corresponding estimates obtained in [14] for the one-dimensional problem ($p = 1$) (cf. [17]).

Besides problems III_k we shall study the standard problems IV_k which differ from III_k only in the expression for

$$\psi_\alpha = (\mu_\alpha)_{\hat{x}_\alpha} + \overset{*}{\psi}_\alpha. \tag{12}$$

It will be shown below that the estimate of the solution of problem III_k reduces to the estimate of the solution of the standard problem IV_k (cf. Para. 5).

2. ENERGY INEQUALITIES ON THE CHAIN $Ц_\alpha$

In deriving energy inequalities we shall use the following systematic device. We consider an arbitrary chain $Ц_\alpha$ of the fixed direction x_α on which, for $t = t_{j+\alpha/p}$ we write down the n-th rank energy identity. Using majorant estimates [14] we obtain energy inequalities which we then weight-sum

$$H_\alpha = \prod_{k\neq\alpha}^{1-p} \hbar_k \tag{13}$$

over all the chains $Ц_\alpha$ for fixed α. As a result we obtain inequalities on the net ω_h.

Thus, let us consider the arbitrary chain $Ц_\alpha$ with the boundary points $x^{(-\alpha)} \in \gamma_\alpha^-$ and $x^{(+\alpha)} \in \gamma_\alpha^+$. Let $ц_\alpha$ denote the set of internal nodes of this chain $ц_\alpha^+ = ц_\alpha + x^{(+\alpha)}$, $ц_\alpha^- = ц_\alpha + x^{(-\alpha)}$, $\bar{ц}_\alpha = ц_\alpha + x^{(+\alpha)} + x^{(-\alpha)}$.

Let v and z be certain net functions given on $\bar{\omega}_h$. We introduce scalar products and norms:

$$\begin{gathered}
(v, z)^*_{ц_\alpha} = \sum_{x\in ц_\alpha} vz\hbar_\alpha, \qquad (v, z)_{ц_\alpha} = \sum_{x\in ц_\alpha} vzh_\alpha, \qquad (v, z)^+_{ц_\alpha} = \sum_{x\in ц_\alpha} vzh_{\alpha+}, \\
(v, z]_{ц_\alpha} = \sum_{x\in ц_\alpha^+} vzh_\alpha, \qquad [v, z)_{ц_\alpha} = \sum_{x\in ц_\alpha^-} vzh_{\alpha+}, \\
\|v\|_{2\alpha,1} = (v, v)^{*1/2}_{ц_\alpha}, \qquad \|z_{\bar{x}_\alpha}\|_{2_\alpha,1} = (1, z^2_{\bar{x}_\alpha}]^{1/2}_{ц_\alpha}.
\end{gathered} \tag{14}$$

Let us consider the difference operator $\Lambda_\alpha^0 z = (a_\alpha z_{\bar{x}_\alpha})_{\hat{x}_\alpha}$. The first difference formula of Green will obviously have the form

$$(\Lambda_\alpha^0 z, z)^*_{ц_\alpha} = -(a_\alpha, z^2_{\bar{x}_\alpha}]_{ц_\alpha} + a_\alpha z_{\bar{x}_\alpha} z\big|_{x=x^{(+\alpha)}} - a_\alpha^{(+1_\alpha)} z_{x_\alpha} z\big|_{x=x^{(-\alpha)}}. \tag{15}$$

If $z = 0$ when $x = x^{(\pm\alpha)}$, the substitution becomes zero. Suppose that z satisfies the homogeneous boundary condition

$$z = \beta_\alpha^\pm z^{(\mp 1_\alpha)} \text{ for } x = x^{(\pm\alpha)}. \tag{16}$$

Using the fact that $z^{(+1_\alpha)} = z + h_{\alpha+} z_{x_\alpha}$, $z^{(-1_\alpha)} = z - h_\alpha z_{\bar{x}_\alpha}$, we obtain from (16)

$$(\Lambda_\alpha^0 z, z)^{\bullet}_{u_\alpha} = - I_{1,1}^{(\alpha)}, \qquad I_{1,1}^{(\alpha)} = (a_\alpha, z_{\bar{x}_\alpha}^2]_{u_\alpha} + \tag{17}$$
$$+ h_{\alpha+} \varkappa_\alpha^- a_\alpha^{(+1_\alpha)} z_{x_\alpha}^2 \Big|_{x=x^{(-\alpha)}} + h_\alpha \varkappa_\alpha^+ a_\alpha z_{\bar{x}_\alpha}^2 \Big|_{x=x^{(+\alpha)}}.$$

As usual we shall use Hölder's inequality as well as Green's difference formula and the elementary inequalities

$$\prod_{k=1}^m x_k^{\mu_k} \leqslant \sum_{k=1}^m \mu_k x_k, \qquad x_k > 0, \qquad \mu_k > 0, \qquad \sum_{k=1}^m \mu_k = 1, \tag{18}$$

$$|ab| \leqslant \frac{c_0}{2} a^2 + \frac{1}{2c_0} b^2 \quad (c_0 > 0 - \text{an arbitrary constant}). \tag{19}$$

Positive constants which depend on $c_1, \ldots, c_6$ and on the diameter of the region G and do not depend on the net will be denoted by M and expressions for them will not, as a rule, be given.

It is not difficult to see that the following lemma holds.

Lemma 1. If z satisfies the condition

$$z = \beta_\alpha^\pm (z^{(\mp 1_\alpha)} + \nu_\alpha^\pm), \qquad x = x^{(\pm\alpha)} \in \gamma_\alpha^\pm, \qquad 0 \leqslant \beta_\alpha^\pm \leqslant \beta^* < 1 \tag{20}$$

(β^* is a constant which does not depend on the net) then

$$\|z\|_{0,u_\alpha}^2 = \max_{x \in \bar{u}_\alpha} |z|^2 \leqslant M_0 (a_\alpha, z_{\bar{x}_\alpha}^2]_{u_\alpha} + M_0' |\bar{\nu}_\alpha|^2, \tag{21}$$

$$\|\overset{n}{z}\|_{0,u_\alpha} \leqslant M_1 \left(a_\alpha, \left(\overset{n-1}{z}_{\bar{x}_\alpha}\right)^2\right]_{u_\alpha} + (M_1' 2^n |\bar{\nu}_\alpha|)^{2^n}, \tag{22}$$

where $\overset{n}{z} = z^{2^n}$, M_0, M_0', M_1, M_1' are positive constants which depend on c_1, β^* and the length l_α of the chain $\amalg_\alpha$, and $|\bar{\nu}_\alpha|$ is either of the quantities $|\nu_\alpha^\pm|$.

It follows from condition (20) that $|z| \leqslant \beta^* |z^{(+1_\alpha)}| + \beta^* |\nu_\alpha^-|$ $(x \in \gamma_\alpha^-)$. The inequality $|z^{(+1_\alpha)}| \leqslant |z(x^{(-\alpha)}, t)| + h_{\alpha+}^{1/2} c_1^{-1/2} I^{1/2}$ gives $|z^{(+1_\alpha)}| \leqslant \beta^* (1 - \beta^*)^{-1} |\nu_\alpha^-| + h_{\alpha+}^{1/2} c_1^{-1/2} (1 - \beta^*)^{-1} I^{1/2}$ and $|z(x^{(-\alpha)}, t)| \leqslant (\beta^*)^2 (1 - \beta^*)^{-1} |\nu_\alpha^-| + h_{\alpha+}^{1/2} \beta^* c_1^{-1/2} \times (1 - \beta^*)^{-1} I^{1/2}$, where $I = (a_\alpha, z_{\bar{x}_\alpha}^2]_{u_\alpha}$. From this and from the inequality $\|z\|_{0,u_\alpha} \leqslant |z(x^{(-\alpha)}, t)| + l_\alpha^{1/2} c_1^{-1/2} I^{1/2}$ we have (21). On the net $\omega_h^{(2)}$ $\beta_\alpha^\pm = 0$ and

$$\| z \|_{0,u_\alpha} \leqslant M\,(a_\alpha,\ z^2_{\bar{x}_\alpha}]^{1/2}_{u_\alpha}, \qquad \| \overset{n}{z} \|_{0,u_\alpha} \leqslant M\,(a_\alpha,\ \Big(\overset{n-1}{z}_{\bar{x}_\alpha}\Big)^2\,]_{u_\alpha}. \tag{23}$$

By analogy with [1] let us write down at once the n-th rank energy identity on the chain Π_α:

$$((\rho,\overset{n}{z})^{*}_{u_\alpha})_{\bar{t}_\alpha} + 2I^{(\alpha)}_{n,1} + P^{(\alpha)}_{n,1} + 2\,[d_\alpha,\overset{n}{z}]_{u_\alpha} = 2^n\,(Q_\alpha(z) + \Psi_\alpha,\ z^{\sigma_n})^{*}_{u_\alpha} + \tag{24}$$

$$+ (\rho_{\bar{t}_\alpha},\ \check{\overset{n}{z}})^{*}_{u_\alpha} + R^{(\alpha)}_{n,1},$$

$$I^{(\alpha)}_{n,1} = (a_\alpha,\ \Big(\overset{n-1}{z}_{\bar{x}_\alpha}\Big)^2\,]_{u_\alpha} + \sum_{k=0}^{n-2} 2^{n-k-2}\,\{(a_\alpha\Big(\overset{k}{z}_{\bar{x}_\alpha}\Big)^2,\ z^{\sigma_n-\sigma_{k+1}}]_{u_\alpha} + \tag{25}$$

$$+ [a_\alpha^{(+1_\alpha)}\Big(\overset{k}{z}_{x_\alpha}\Big)^2,\ z^{\sigma_n-\sigma_{k+1}})^{+}_{u_\alpha}\},$$

$$P^{(\alpha)}_{n,1} = \tau \sum_{k=0}^{n-1} 2^{n-k-1}\,(\rho\,\Big(\overset{k}{z}_{\bar{t}_\alpha}\Big)^2,\ z^{\sigma_n-\sigma_{k+1}})^{*}_{u_\alpha}, \qquad \sigma_k = 2^k - 1, \tag{26}$$

$$R^{(\alpha)}_{n,1} = 2^n\,\{\nu^{+}_\alpha a_\alpha z^{\sigma_n} h^{-1}_\alpha \big|_{x=x^{(+\alpha)}} - \nu^{-}_\alpha a_\alpha^{(+1_\alpha)}\, z^{\sigma_n} h^{-1}_{\alpha+} \big|_{x=x^{(-\alpha)}}\}, \tag{27}$$

$$[d_\alpha,\overset{n}{z}]_{u_\alpha} = (d_\alpha,\overset{n}{z})^{*}_{u_\alpha} + a_\alpha \overset{n}{z}\,(\varkappa^{+}_\alpha h_\alpha)^{-1}\big|_{x=x^{(+\alpha)}} + a_\alpha^{(+1_\alpha)}\overset{n}{z}\,(\varkappa^{-}_\alpha h_{\alpha+})^{-1}\big|_{x=x^{(-\alpha)}}\}. \tag{28}$$

When deriving identity (24) we wrote the boundary conditions (2) in the form of conditions of the third kind:

$$z_{x_\alpha} = z/\varkappa^{-}_\alpha h_{\alpha+} - \nu^{-}_\alpha/h_{\alpha+} \text{ for } x = x^{(-\alpha)}; \quad -z_{\bar{x}_\alpha} = z/\varkappa^{+}_\alpha h_\alpha - \nu^{+}_\alpha/h_\alpha \text{ for } x = x^{(+\alpha)}$$

and used an expression for

$$\overset{n}{z}_{x_\alpha} = 2^n z^{\sigma_n} z_{x_\alpha} - \sum_{k=0}^{n-1} 2^{n-k-1} h_{\alpha+}\,(\overset{k}{z}_{x_\alpha})^2\, z^{\sigma_n-\sigma_{k+1}} \tag{29}$$

and a similar expression for $\overset{n}{z}_{\bar{x}_\alpha}$. It must be borne in mind that $\overset{n}{z}(\varkappa_\alpha h_\alpha)^{-1} = 0$, if $\kappa_\alpha = 0$. On the net $\omega^{(2)}_h\,[d_\alpha,\overset{n}{z}]_{u_\alpha} = (d_\alpha,\overset{n}{z})^{*}_{u_\alpha}$ and $R^{(\alpha)}_{n,1} = 0$, if $z|_{\gamma_\alpha} = 0$. If $z = \nu^{\pm}_\alpha$ when $x \in \gamma^{\pm}_\alpha$, then $[d_n,\overset{n}{z}]_{u_\alpha} = (d_\alpha,\overset{n}{z})^{*}_{u_\alpha}$, and formula (27) for $R^{(\alpha)}_{n,1}$ is true.

Let us now suppose that $\nu^{\pm}_\alpha = 0$, i.e. let us consider the problem III°. Putting $n = 1$ in (24) we obtain an identity of the first rank and, according to (17) we can write this in the form

$$((\rho, z^2)^*_{u_\alpha})_{\bar t_\alpha} + 2I^{(\alpha)}_{1,1} + P^{(\alpha)}_{1,1} + 2(d_\alpha, z^2)^*_{u_\alpha} = 2(Q_\alpha(z) + \Psi_\alpha, z)^*_{u_\alpha} + (\rho_{\bar t_\alpha}, \check z^2)^*_{u_\alpha}, \tag{30}$$

$$I^{(\alpha)}_{1,1} = (a_\alpha, z^2_{\bar x_\alpha}]_{u_\alpha} + h_\alpha \varkappa^+_\alpha a_\alpha z^2_{\bar x_\alpha}\Big|_{x=x^{(+\alpha)}} + h_{\alpha+}\varkappa^-_\alpha a^{(+1_\alpha)}_\alpha z^2_{x_\alpha}\Big|_{x=x^{(-\alpha)}}, \tag{31}$$

$$P^{(\alpha)}_{1,1} = \tau(\rho, z^2_{\bar t_\alpha})^*_{u_\alpha}.$$

Let us consider problem III_2, of which problem III_1 is a special case, for $g^\pm_{\alpha k} = 0$, $k = 1, 2$. We introduce the function η_α, putting

$$(\eta_\alpha)_{\hat x_\alpha} = \psi_\alpha, \quad \eta^{(+1_\alpha)}_\alpha = 0 \quad \text{for } x = x^{(-\alpha)}, \tag{32}$$

and put

$$\|\psi_\alpha\|_{4_\alpha,1} = \|\eta_\alpha\|_{2_\alpha,1} + |\eta_\alpha(x^{(+\alpha)}, t)| = \|\eta_\alpha\|_{\bar 2_\alpha,1}, \qquad \|\eta_\alpha\|_{2_\alpha,1} = (1, \eta^2_\alpha]^{1/2}_{u_\alpha}. \tag{33}$$

Lemma 2. If z satisfies the homogeneous boundary conditions (16), then

$$2(\psi_\alpha, z)^*_{u_\alpha} \leqslant \tfrac{1}{8} I^{(\alpha)}_{1,1} + M\|\psi_\alpha\|^2_{4_\alpha,1}. \tag{34}$$

For

$$2(\psi_\alpha, z)^* = -2(\eta_\alpha, z_{\bar x_\alpha}]_{u_\alpha} + 2\eta_\alpha z\big|_{x=x^{(+\alpha)}} \leqslant$$

$$\leqslant 2\|\eta_\alpha\|_{2_\alpha,1}\|z_{\bar x_\alpha}\|_{2_\alpha,1} + 2|\eta_\alpha(x^{(+\alpha)}, t)||z(x^{(+\alpha)}, t)|.$$

Then using Lemma 1 we obtain (34).

Lemma 3. We have the estimate

$$2(Q^{(1)}_\alpha(z) + (g^+_{\alpha 1} z)_{\hat x_\alpha} + (g^-_{\alpha 1} z)_{\check x_\alpha}, z)^*_{u_\alpha} \leqslant \tfrac{1}{4} I^{(\alpha)}_{1,1} + M\|z\|^2_{2_\alpha,1} + M'(\check\rho, \check z^2)^*_{u_\alpha}. \tag{35}$$

Lemma 3 can be proved by analogy with [14], using Lemma 1, Green's difference formula and inequality (19).

Lemma 4. On the net $\omega^{(2)}_h$ we have the estimate

$$2((g^+_{\alpha 2}\check z)_{\hat x_\alpha} + (g^-_{\alpha 2}\check z)_{\check x_\alpha}, z)^*_{u_\alpha} \leqslant \tfrac{1}{8} I^{(\alpha)}_{1,1} + M(\check\rho, \check z^2)^*_{u_\alpha}. \tag{36}$$

The inequality (36) follows from the estimate of the form

$$2\left((g^{+}_{\alpha 2}\check{z})_{\hat{x}_\alpha},\ z\right)^{\bullet}_{u_\alpha} = -2\left(g^{+}_{\alpha 2}\check{z},\ z_{\bar{x}_\alpha}\right)_{u_\alpha} \leqslant 2c_3 \|\check{z}\|_{3_\alpha,1} \|z_{\bar{x}_\alpha}\|_{2\alpha,1}.$$

Using Lemmas 2-4 we obtain an inequality of the first rank on the chain $Ц_\alpha$:

$$(\rho,\ z^2)^{\bullet}_{u_\alpha} + \tau I^{(\alpha)}_{1,1} + \tau P^{(\alpha)}_{1,1} + 2c^{*}\tau\,(1,\ z^2)^{\bullet}_{u_\alpha} \leqslant (1 + M\tau)\,(\check{\rho},\ \check{z}^2)^{\bullet}_{u_\alpha} + \tag{37}$$
$$+ M\tau\,\|\psi_\alpha\|^2_{4_\alpha,1} + 2\,(\mathring{\psi}_\alpha,\ z)^{\bullet}_{u_\alpha}\times\tau,$$

where c^* is an arbitrary positive constant depending on the choice of $\overline{M}$. It is not difficult, by analogy with [14], [15], [17] to obtain an n-th rank energy inequality on the chain $Ц_\alpha$:

$$(\rho,\ \overset{n}{z})^{\bullet}_{u_\alpha} + \tau I^{(\alpha)}_{n,1} + \tau P^{(\alpha)}_{n,1} + 2^{m+1}\tau c^{*}\,(1,\ \overset{n}{z})^{\bullet}_{u_\alpha} \leqslant (1 + M\tau)\,(\check{\rho},\ \check{\overset{n}{z}})^{\bullet}_{u_\alpha} + \tag{38}$$
$$+ \tau\,(M_n\|\psi_\alpha\|_{4_\alpha,1})^{2n} + 2^n\tau\,(\mathring{\psi}_\alpha,\ z^{\sigma n})^{\bullet}_{u_\alpha},\quad m = n \text{ for } \mathrm{III}_1,\quad m = 2n \text{ for } \mathrm{III}_2,$$

where $M_n = M \times 2^n$. Conditions (7) and (7′) are satisfied.

The last terms in (37) and (38) are estimated below.

3. ENERGY INEQUALITIES ON THE NET ω_h

Let us consider the sums (14) with respect to $Ц_\alpha$. Multiplying them with H_α and summing over all chains of the given direction α we obtain sums over ω_h:

$$(v,\ z) = \sum_{\omega_h} vzH, \qquad (v,\ z)_\alpha = \sum_{\omega_h} vzH^{(\alpha)},$$

$$(v,\ z)^{+}_{\alpha} = \sum_{\omega_h} vzH^{(+\alpha)}, \qquad (v,\ z]_\alpha = \sum_{\omega_h^{+\alpha}} vzH^{(\alpha)},$$

$$[v,\ z)^{+}_{\alpha} = \sum_{\omega_h^{-\alpha}} vzH^{(+\alpha)}, \qquad H^{(\alpha)} = H_\alpha h_\alpha, \qquad H^{(+\alpha)} = H_\alpha h_{\alpha+},$$

$$H = H_\alpha \hbar_\alpha = \prod_{k=1}^{p} \hbar_k, \quad \omega_h^{\pm\alpha} = \omega_h + \gamma_\alpha^{\pm}.$$

These sums are associated with the norms $\|v\| = (v,\ v)^{1/2}$, $\|v\|_{2_\alpha} = (v,\ v)^{1/2}_\alpha$ or $\|v\|_{2_\alpha} = (v,\ v]^{1/2}_\alpha$, for example

$$\| v_{\bar{x}_\alpha} \|_{2\alpha} = (v_{\bar{x}_\alpha},\ v_{\bar{x}_\alpha}]_\alpha^{1/2}. \tag{39}$$

In addition, we shall need the norms

$$\| v \|_0 = \max_{\omega_h} | v(x, t) |, \qquad \overline{\| v(x, t_{j+1}) \|} = \left(\sum_{j'=1}^{j+1} \| v(x, t_{j'}) \|^2 \tau_{j'} \right)^{1/2}. \tag{40}$$

Now let us multiply (37) and (38) by H_α and sum over all the chains of the direction α. As a result we obtain energy inequalities of the first and n-th rank on the net

$$(\rho, z^2) + \tau (I_1^{(\alpha)} + P_1^{(\alpha)}) + 2\tau c^* (1, z^2) \leqslant \tag{41}$$
$$\leqslant (1 + M\tau)(\check{\rho}, \check{z}^2) + \tau M \| \psi_\alpha \|^2_{4\alpha} + 2 (\mathring{\psi}_\alpha, z)\tau,$$

$$(\rho, \overset{n}{z}) + \tau (I_n^{(\alpha)} + P_n^{(\alpha)}) + \tau 2^{m+1} c^* (1, \overset{n}{z}) \leqslant (1 + M\tau)(\check{\rho}, \check{\overset{n}{z}}) + \tag{42}$$
$$+ \tau (M \times 2^n \| \psi_\alpha \|_{4\alpha, 0})^{2n} + 2^n (\mathring{\psi}_\alpha, z^{\sigma_n})\tau.$$

We have used the notation

$$\| \psi_\alpha \|_{4\alpha} = \| \eta_\alpha \|_{2\alpha} + \| \eta_\alpha \|_{2, \gamma_\alpha^+}, \qquad \| \eta_\alpha \|_{2, \gamma_\alpha^+} = \left(\sum_{\gamma_\alpha^+} \eta_\alpha^2 H_\alpha \right)^{1/2}, \tag{43}$$

$\| \psi_\alpha \|_{4_{\alpha, 0}}$ is the maximum value of $\| \psi_\alpha \|_{4\alpha, 1}$ over the whole net ω_h, or, more exactly, over all the chains of the given direction α, $\| \psi_\alpha \|_{4\alpha, 0} = \| \| \psi_\alpha \|_{4\alpha, 1} \|_0$

$$I_n^{(\alpha)} = (a_\alpha,\ \left(\overset{n-1}{z_{\bar{x}_\alpha}}\right)^2] + \sum_{k=0}^{n-2} 2^{n-k-2} \left\{ (a_\alpha \left(\overset{k}{z_{\bar{x}_\alpha}}\right)^2,\ z^{\sigma_n - \sigma_{k+1}}]_\alpha + \right. \tag{44}$$
$$\left. + [a_\alpha^{(+1_\alpha)} \left(\overset{k}{z_{x_\alpha}}\right)^2,\ z^{\sigma_n - \sigma_{k+1}})_\alpha^+ \right\},$$

$$P_n^{(\alpha)} = \tau \sum_{k=0}^{n-1} 2^{n-k-1} (\rho \left(\overset{k}{z_{\bar{t}_\alpha}}\right)^2,\ z^{\sigma_n - \sigma_{k+1}}), \qquad \alpha = 1, 2, \ldots, p, \quad n = 1, 2, \ldots . \tag{45}$$

Let us recall that (42) and (41) were obtained for sufficiently small $\tau \leqslant \tau_0$ and for arbitrary n. Summation of (41) and (42) over $\alpha = 1, 2, \ldots, p$ gives

$$(\rho, z^2)^{j+1} + \tau (I_1 + P_1) + \tau c^* \sum_{\alpha=1}^{p} (1, z^2)^{j+\alpha/p} \leqslant (1 + M\tau)(\rho, z^2)^j + \tag{46}$$

$$+\tau M\|\psi\|_4^2+2\sum_{\alpha=1}^{p}(\overset{\circ}{\psi}_\alpha, z^{j+\alpha/p})\,\tau,$$

$$(\rho, \overset{n}{z})^{j+1}+\tau(I_n+P_n)+2^m\tau c^*\sum_{\alpha=1}^{p}\left(1, \overset{n}{z}\right)^{j+\alpha/p}\leqslant(1+M\tau)(\rho, \overset{n}{z})^j+ \tag{47}$$

$$+\tau(M\times 2^n\|\psi\|_{4,0})^{2n}+2^n\sum_{\alpha=1}^{p}(\overset{\circ}{\psi}_\alpha, (z^{\sigma_n})^{j+\alpha/p})\,\tau,$$

$$I_n=I_n^{j+1}=\sum_{\alpha=1}^{p}I_n^{(\alpha)},\quad P_n=P_n^{j+1}=\sum_{\alpha=1}^{p}P_n^{(\alpha)},\quad m=n \text{ for III}_1,\quad m=2n \text{ for III}_2,$$

$$\|\psi\|_4=\sum_{\alpha=1}^{p}\|\psi_\alpha\|_{4_\alpha},\qquad \|\psi\|_{4,0}=\sum_{\alpha=1}^{p}\|\psi_\alpha\|_{4_\alpha,0},\qquad \|\psi_\alpha\|_{4_\alpha,0}=\|\|\psi_\alpha\|_{4_\alpha,1}\|_0, \tag{48}$$

$\|\psi_\alpha\|_{4_\alpha}$ and $\|\psi_\alpha\|_{4_\alpha,1}$ are defined by formulae (43) and (33).

Let us now estimate the last terms in (46) and (47).

4. BASIC LEMMA

Lemma 5. Suppose that the arbitrary net function $z = z(x, t)$, given on $\bar{\Omega}^k$ $(k = 1, 2)$ satisfies conditions (16), and $\overset{\circ}{\psi}_\alpha$ satisfies the condition

$$\sum_{\alpha=1}^{p}\overset{\circ}{\psi}_\alpha=0. \tag{49}$$

Then

$$2^n\sum_{\alpha=1}^{p}(\overset{\circ}{\psi}_\alpha, (z^{\sigma_n})^{j+\alpha/p})\leqslant\tfrac{1}{2}(I_n+P_n)+M\sum_{\alpha=1}^{p-1}(1, \overset{n}{z})^{j+\alpha/p}+ \tag{50}$$

$$+\left(M2^n\sqrt{\tau}\sum_{\alpha=1}^{p-1}\|\overset{\circ}{\psi}_\alpha\|_0\right)^{2n},$$

$$2\sum_{\alpha=1}^{p}(\overset{\circ}{\psi}_\alpha, z^{j+\alpha/p})\leqslant\tfrac{1}{2}P_1+M\tau\sum_{\alpha=1}^{p-1}\|\overset{\circ}{\psi}_\alpha\|^2. \tag{51}$$

If ω_τ is a normal net $(\tau \leqslant m^* \check{\tau})$, then

$$2^n \sum_{\alpha=1}^{p} (\overset{\circ}{\psi}_\alpha, (z^{\sigma_n})^{j+\alpha/p}) \leqslant \frac{1}{2}(I_n + P_n) + \frac{1}{2}\frac{\check{\tau}}{\tau}\check{I}_n + (M 2^n \sqrt{\tau}\, \| \overset{\circ}{\psi} \|_{2,0})^{2n}, \quad (52)$$

where

$$\| \overset{\circ}{\psi} \|_{2,0} = \sum_{\alpha=1}^{p} \| \overset{\circ}{\psi}_\alpha \|_{2_\alpha,0}, \qquad \| \overset{\circ}{\psi}_\alpha \|_{2_\alpha,0} = \| \| \overset{\circ}{\psi}_\alpha \|_{2_\alpha,1} \|_0,$$

$$I_n = I_n^{j+1}, \qquad \check{I}_n = I_n^j, \qquad \tau = \tau_{j+1}, \qquad \check{\tau} = \tau_j. \quad (53)$$

Let us just derive inequality (50). Putting $v^{j+\alpha/p}$ in the form

$$v^{j+\alpha/p} = v^{j+1} - \tau \sum_{m=\alpha+1}^{p} v_{\bar{t}_m},$$

where $v = z^{\sigma_n}$, and using the formula

$$(z^{\sigma_n})_{\bar{t}_m} = \sum_{k=0}^{n-1} \check{z}^{\sigma_k} z^{\sigma_{n-}\sigma_{k+1}} \overset{k}{z}_{\bar{t}_m}$$

(cf. [14]) and condition (49) after changing the order of summation over α and m we obtain

$$2^n \sum_{\alpha=1}^{p} (\overset{\circ}{\psi}_\alpha, z^{\sigma_n}) = -2^n \tau \sum_{k=0}^{n-1} \sum_{m=2}^{p} \left(\check{z}^{\sigma_k} z^{\sigma_{n-}\sigma_{k+1}} \overset{k}{z}_{\bar{t}_m}, \sum_{\alpha=1}^{m-1} \overset{\circ}{\psi}_\alpha\right),$$

where the summation sign on the right contains $z = z^{j+m/p}$, $\check{z} = z^{j+(m-1)/p}$. Then arguing as in [14] first for the chain Π_α and using Lemma 1 we find

$$2^n \tau \sum_{k=0}^{n-1} (\check{z}^{\sigma_k} z^{\sigma_{n-}\sigma_{k+1}} z_{\bar{t}_m}, \overset{\circ}{\psi}_\alpha) \leqslant c_0 (I_n + P_n) + M (1, \overset{n}{z})^{j+(m-1)/p} + (M 2^n \sqrt{\tau}\, \| \overset{\circ}{\psi}_\alpha \|_0)^{2n}.$$

This and the previous formula give (50).

If $\overset{\circ}{\psi}_\alpha$ has the form

$$\overset{\circ}{\psi}_\alpha = \sum_{\beta=1}^{p} \overset{\circ}{\psi}_{\alpha\beta}, \quad (54)$$

then in (53) we must put

$$\|\mathring{\psi}\|_{2,0} = \sum_{\alpha,\beta=1}^{p} \|\mathring{\psi}_{\alpha\beta}\|_{2_\beta,0}. \tag{55}$$

5. *A PRIORI* ESTIMATES IN THE MEAN

Let us now return to the first rank energy inequality (46). Substituting here the estimate (51) we obtain

$$(\rho, z^2)^{j+1} + \tau I_1 \leqslant (1 + M\tau)(\rho, z^2)^j + M\tau \|\Psi^{j+1}\|^2, \qquad \tau = \tau_{j+1}, \tag{56}$$

$$\|\Psi\|^2 = \|\psi\|_4^2 + \tau\|\mathring{\psi}\|^2, \qquad \|\psi\|_4 = \sum_{\alpha=1}^{p} \|\psi_\alpha\|_{4_\alpha}, \qquad \|\mathring{\psi}\| = \sum_{\alpha=1}^{p} \|\mathring{\psi}_\alpha\|, \tag{57}$$

where $\|\psi_\alpha\|_{4_\alpha}$ is given by formula (43).

Since $z(x, 0) = 0$, Lemma 4_* of [14] gives

$$(\rho, z^2)^{j+1} + \sum_{j'=1}^{j+1} \tau_{j'} I_1^{j'} \leqslant M \sum_{j'=1}^{j+1} \tau_{j'} \|\Psi^{j'}\|^2 = M\overline{\|\Psi(x, t_{j+1})\|}^2. \tag{58}$$

This proves the following theorem.

Theorem 1. The solution of problems III_k° and IV_k°, $k = 1, 2$ on any sequence of non-uniform nets $\Omega^{(1)}$ and $\Omega^{(2)}$ for sufficiently small $\tau < \tau_0$ satisfies the inequality

$$\|z(x, t_{j+1})\| + \sqrt{\tau_{j+1}}\|z_{\bar{x}}(x, t_{j+1})\| \leqslant M\overline{\|\Psi(x, t_{j+1})\|}, \tag{59}$$

where $\|\Psi\|$ is defined by formulae (57) and (43) and for problem IV_k°

$$\|\psi_\alpha\|_{4_\alpha} = \|\overset{*}{\psi}_\alpha\|_{4_\alpha} + \|\mu_\alpha\|_{2_\alpha} + \|\mu_\alpha\|_{2,\gamma_\alpha^+} + \|\mu_\alpha\|_{2,\gamma_\alpha^-}, \tag{60}$$

$$\|z_{\bar{x}}(x, t_{j+1})\| = \sum_{\alpha=1}^{p} \|z_{\bar{x}_\alpha}(x, t_{j+\alpha/p})\|_{2_\alpha}.$$

Corollary. If $\psi_\alpha = 0$ then for sufficiently small $\|\tau_0\| < \tau_0$ the solution of problem III_2° has the estimate

$$\|z(x, t_{j+1})\| + \sqrt{\tau_{j+1}}\|z_{\bar{x}}(x, t_{j+1})\| \leqslant M\left(\sum_{j'=1}^{j+1} \tau_{j'}^2 \|\mathring{\psi}^{j'}\|^2\right)^{1/2}, \tag{61}$$

where $\| \overset{\circ}{\psi} \|$ is given by formula (57).

We recall that we have agreed to consider only those problems III_k for which $g^{\pm}_{\alpha 2} = 0$ on $\Omega^{(1)}$.

According to (5), the function ψ_α for problem III_k ($k = 1, 2$) has the form

$$\psi_\alpha = \sum_{\beta=1}^{p} [(\mu_{\alpha\beta})_{\hat{x}_\beta} + \psi^*_{\alpha\beta}]. \tag{62}$$

It is then clear that the norm (43) is not suitable for the estimate (62) on a non-uniform net since it will contain the step ratio $\hbar_\alpha/\hbar_\beta$. Therefore we first reduce problem III_k to problem IV_k for which

$$\psi_\alpha = (\mu_\alpha)_{\hat{x}_\alpha} + \psi^*_\alpha, \tag{63}$$

and then use the norm (60).

It was shown in Section 1 that

$$\mu_{\alpha\beta} = \mu^0_{\alpha\beta} h^2_\beta, \text{ where } \mu^0_{\alpha\beta} = O(1). \tag{64}$$

Let us introduce the notation

$$\|\mu^0\|_0 = \sum_{\alpha,\,\beta=1}^{p} \|\mu^0{}_{\alpha\beta}\|_0, \qquad \|\overline{\mu^0}\|_0 = \max_{\omega_\tau} \|\mu^0\|_0. \tag{65}$$

Lemma 6. Let z be the solution of problem III_2 and v the solution of the same problem with the right hand side

$$\Psi_\alpha = \overset{\circ}{\psi}_\alpha + (\mu_\alpha)_{\hat{x}_\alpha} + \psi^*_\alpha, \text{ where } \mu_\alpha = \sum_{\beta=1}^{p} \mu_{\beta\alpha}, \qquad \psi^*_\alpha = \sum_{\beta=1}^{p} \psi^*_{\alpha\beta}; \tag{66}$$

then for sufficiently small $\tau < \tau_0$ we have the estimate

$$\| z(x, t_{j+1}) - v(x, t_{j+1}) \| \leqslant M \|\overline{\mu^0}\|_0 (\|\tau\|_2 + \|\hbar\|^2), \tag{67}$$

where

$$\|\tau\|_2 = \|\tau\|_2^{j+1} = \left(\sum_{j'=1}^{j+1} \tau^3_{j'} \right)^{1/2}, \qquad \|\hbar\| = \sum_{\beta=1}^{p} \|\hbar_\beta\|. \tag{68}$$

We recall that in our terminology (cf. Para. 1) v is the solution of problem IV_2. Consider the difference $w = z - v$. For this difference we obtain the conditions

$$\left.\begin{aligned} \rho w_{\bar{t}_\alpha} - \Lambda_\alpha w + d_\alpha w = Q_\alpha^{(2)}(w) + \widetilde{\psi}_\alpha, \qquad \widetilde{\psi}_\alpha = \sum_{\beta=1}^{p} (\mu_{\alpha\beta})_{\hat{x}_\beta} - \sum_{\beta=1}^{p} (\mu_{\beta\alpha})_{\hat{x}_\alpha} \\ w = \beta_\alpha^{\pm} w^{(\mp 1_\alpha)} \text{ for } x \in \gamma_\alpha^{\pm}; \qquad w(x, 0) = 0 \text{ for } x \in \overline{\omega}_h. \end{aligned}\right\} \quad (69)$$

It is clear from this that

$$\sum_{\alpha=1}^{p} \widetilde{\psi}_\alpha = 0 \tag{70}$$

and, therefore, we can use Lemma 5. Using the fact that

$$(\mu_{\alpha\beta})_{\hat{x}_\beta} \leqslant 2\,\{h_\beta |\mu^0_{\alpha\beta}| + h_{\beta+} |(\mu^0_{\alpha\beta})^{(+1_\beta)}|\,, \quad h_\beta \leqslant 2\hbar_\beta, \quad h_{\beta+} \leqslant 2\hbar_\beta,$$

we obtain $\|\widetilde{\psi}_\alpha\|^2 \leqslant M\|\hbar\|^2$, $\|\widetilde{\psi}\|^2 \leqslant M\|\hbar\|^2$. Then using the estimate (61) for w and the fact that $2\|\tau\|_2^{1/2}\|\hbar\| \leqslant \|\tau\|_2 + \|\hbar\|^2$, we arrive at inequality (67).

From Theorem 1 and Lemma 6 we have Theorem 2.

Theorem 2. Let $z = z(x, t)$ be the solution of problem III_2^0. Then for sufficiently small $\tau < \tau_0$ on an arbitrary sequence of nets $\Omega^{(1)}$ and $\Omega^{(2)}$ we have the estimate

$$\|z(x, t_{j+1})\| \leqslant M\overline{\|\Psi(x, t_{j+1})\|} + M\|\overline{\mu^0}\|(\|\tau\|_2 + \|\hbar\|^2), \tag{71}$$

where $\|\Psi(x, t)\|$ is the expression defined by formulae (57) and (60) with

$$\mu_\alpha = \sum_{\beta=1}^{p} \mu_{\beta\alpha}, \qquad \psi^*_\alpha = \sum_{\beta=1}^{p} \psi^*_{\alpha\beta}.$$

We note that since III_1 is a special case of the problem III_2 Theorems 1 and 2 also refer to problem III_1.

The maximum principle is valid for problem III_1 and so its solution depends continuously on the boundary data (cf. [10]). In order to obtain an estimate for the solution of problem III_2 for $\nu_\alpha^{\pm} \neq 0$ it is sufficient to estimate the solution of the corresponding problem III_1 with the same data and then use the following lemma.

Lemma 7. Let z be the solution of problem III_2 and v the solution of problem III_1 with the same coefficients and Ψ_α, $\nu_\alpha^{\pm}$. Then for sufficiently small $\|\tau\|_0 < \tau_0$ on any sequence of nets $\Omega^{(1)}$ and $\Omega^{(2)}$ we have the estimate

$$\| z(x, t_{j+1}) - v(x, t_{j+1}) \| \leqslant M \overline{\| v(x, t_{j+1}) \|}. \tag{72}$$

For $w = z - v$ is the solution of problem III_2° with homogeneous boundary conditions and the right-hand side $\tilde{\Psi}_\alpha = Q^*_\alpha(v)$. Applying Theorem 1 to w we obtain (72).

6. UNIFORM ESTIMATES FOR THE SOLUTION OF PROBLEM III_1

Theorem 3. The solution of problem IV_1° on an arbitrary sequence of nets $\Omega^{(1)}$ and $\Omega^{(2)}$ for $\|\tau\|_0 < \tau_0$ has the uniform estimate

$$\| z(x, t_{j+1}) \|_0 \leqslant M \| \Psi(x, t_{j+1}) \|' \ln^\delta \frac{1}{H_*} \text{ for } H_* = \min H \leqslant H_0(\delta), \tag{73}$$

where $\delta > 1$ is an arbitrary number

$$\| \Psi(x, t_{j+1}) \|' = \max_{0 < j' \leqslant j+1} \| \Psi^{j'} \|,$$

$$\| \Psi \| = \| \psi \|_{4,0} + \sqrt{\tau} \| \overset{\circ}{\psi} \|_0.$$

$$\| \psi \|_{4,0} = \sum_{\alpha=1}^{p} \| \psi_\alpha \|_{4_\alpha, 0},$$

$$\| \psi_\alpha \|_{4_\alpha, 0} = \| \| \psi_\alpha \|_{4_\alpha, 1} \|_0. \tag{74}$$

$\| \overset{\circ}{\psi} \|_0 = \sum\limits_{\alpha=1}^{p} \| \overset{\circ}{\psi}_\alpha \|_0$, and $\| \psi_\alpha \|_{4_\alpha, 1}$ is given by formula (33).

If ω_τ is a normal net, then

$$\| \Psi \| = \| \Psi \|_* = \| \psi \|_{4,0} + \sqrt{\tau} \| \overset{\circ}{\psi} \|_{2,0},$$

$$\| \overset{\circ}{\psi} \|_{2,0} = \sum_{\alpha=1}^{p} \| \overset{\circ}{\psi}_\alpha \|_{2_\alpha, 0},$$

$$\| \overset{\circ}{\psi}_\alpha \|_{2_\alpha, 0} = \| \| \overset{\circ}{\psi}_\alpha \|_{2_\alpha, 1} \|_0. \tag{74'}$$

Let us put (50) in (47) and use Lemma 4_a of [14]:

$$(\rho, \overset{n}{z})^{j+1} \leqslant \sum_{j'=1}^{j+1} (M 2^n \| \Psi^{j'} \|)^{2^n} \tau_{j'} \leqslant T (M 2^n \| \Psi^{j+1} \|')^{2^n}. \tag{75}$$

It then follows that $\|z(x,\ t_{j+1})\|_0 \leqslant M \times 2^n \|\Psi(x,\ t_{j+1})\|^0 H_*^{-1/2^n}$. Using (8) and (10) and choosing $n = n(H_*)$ as in [14] we obtain (73). To prove the second part of the theorem we must put (52) in (47) and repeat the previous argument.

Corollary. If $\psi_\alpha = 0$ then (73) takes the form

$$\|z(x,\ t_{j+1})\|_0 \leqslant M \|\sqrt{\tau_{j+1}}\, \overset{\circ}{\psi}(x, t_{j+1})\|_0^0 \ln^8 \frac{1}{H_*}$$

$$H_* \leqslant H_0(\delta),\ \|\tau\|_0 < \tau_0. \tag{76}$$

If $\psi_\alpha = 0$ and the net ω_τ is normal $(\tau \leqslant m^* \check{\tau})$, then

$$\|z(x,\ t_{j+1})\|_0 \leqslant M \|\sqrt{\tau_{j+1}}\, \overset{\circ}{\psi}(x, t_{j+1})\|_{2,0}^0 \ln^8 \frac{1}{H_*}$$

$$H_* \leqslant H_0(\delta),\quad \|\tau\|_0 < \tau_0. \tag{76'}$$

Lemma 8. Let z be the solution of problem III_1 and v the solution of the corresponding problem IV_1 with the right-hand side (66). Then on $\Omega^{(1)}$ and $\Omega^{(2)}$ for $H_* \leqslant H_0(\delta)$, $\|\tau\|_0 < \tau_0$ we have the uniform estimate

$$\|z(x,\ t_{j+1}) - v(x,\ t_{j+1})\|_0 \leqslant M \|\overline{\mu^0}\|_0 (\|\tau\|_0 + \|h\|_0^2) \ln^8 \frac{1}{H_*},$$

$$\|h\|_0 = \sum_{\alpha=1}^{p} \|h_\alpha\|_0. \tag{77}$$

To prove (77) it is sufficient to estimate the solution of problem (69) where $Q_\alpha = Q_\alpha^{(1)}$, using (76).

Let us now pass to problem III_2°. For this problem we have the inequality (75). However, on transforming to the old function z_{old} according to (8) we must multiply z_{new} which figures in all our reasoning by $e^{\overline{M}_0 2^n t_{j+1}}$, and this gives

$$\|z(x,\ t_{j+1})\|_0 \leqslant M e^{M 2^n} \overline{\|\Psi(x_1 t_{j+1})\|}\, H_*^{-1/2^n} \tag{78}$$

for sufficiently small $\|\tau\|_0 \leqslant \tau_0(n) = \tau_0/2^n$, which in this case depends on n. This dependence occurs in connection with the fact that it is not simply M but $\overline{M}/(1 + \overline{M}\tau)$ which enters into d_α (cf. [14]).

Making the requirement that $\overline{M}/(1 + \overline{M}\tau) > M^* 2^n$, and choosing $\overline{M} = \overline{M}_0 2^n$, we obtain $\overline{M}_0 > (1 + \overline{M}_0 2^n \tau) M^*$, i.e. $\overline{M}_0 > M^*/(1 - M^* 2^n \tau)$. It follows that $\tau < \tau_0/2^n$, $\tau_0 < 1/M^*$. It is clear from (78) that $e^{M2^nH_*^{-1/2n}}$ takes its least value if we choose $2^n \sim \sqrt{\ln(1/H_*)}$. This leads to the uniform estimate

$$\|z(x, t_{j+1})\|_0 \leqslant M\overline{\|\Psi(x, t_{j+1})\|}\exp\left(M\sqrt{\ln\frac{1}{H_*}}\right), \tag{79}$$

which is true for sufficiently small $H_* \leqslant H_0$ with the additional condition $\sqrt{\ln(1/H_*)}\,\tau \leqslant \overset{*}{\tau}_0$, where $\overset{*}{\tau}_0$ is a positive constant which does not depend on n, i.e. on H_*. In [17], Section 2 we obtained the estimate (79) for a multi-dimensional scheme, but did not show that there is the relation $\sqrt{\ln(1/H_*)}\,\tau \leqslant \overset{*}{\tau}_0$. between τ and H_*. In [14] we found the estimate (73) for the one-dimensional problem III_2 ($p = 1$). Unfortunately the method of [14] is not suitable for the multi-dimensional case. We note that for the special problem III_2 for $g^{\pm}_{\alpha 1} = 0$, $b^{\pm}_{\alpha} = 0$ on the net $\Omega^{(2)}$ this additional condition is removed.

7. *A PRIORI* ESTIMATES FOR OTHER PROBLEMS

Theorem 4. The solution of the problems III°_{k}, $k = 1', 2', 3$ on any sequence of nets $\Omega^{(1)}$ and $\Omega^{(2)}$ has the estimate (59) for sufficiently small $\|\tau\|_0 < \tau_0$.

The proof is done in a similar way to that of Theorem 1. We have to use the estimate

$$\sum_{\alpha=1}^{p}\left(b^{+}_{\alpha} z^{j+\alpha/p}_{\hat{x}_\alpha} + b^{-}_{\alpha} z^{j+\alpha/p}_{\check{x}_\alpha},\ z^{j+1}\right) \leqslant M\|z^{j+1}\|^2 + c_0 I_1,$$

for problem III_3, where c_0 is an arbitrary number, and for $\mathrm{III}_{1'}$ and $\mathrm{III}_{2'}$ we need the estimate

$$\left(b^{+}_{\alpha-1}\check{z}_{\hat{x}_{\alpha-1}} + b^{-}_{\alpha-1}\check{z}_{\check{x}_{\alpha-1}}, z\right) \leqslant M\|z^{j+\alpha/p}\|^2 + c_0 I_1^{(\alpha-1)}.$$

Lemma 5 does not depend on the form of Q_α and so is true in this case also.

From (59) we have the estimate (61) and using this estimate we can prove Lemma 6 for the given problems.

Problems III_3 and $\mathrm{III}_{2'}$ reduce to problem III_2 and $\mathrm{III}_{1'}$ to problem III_1.

Lemma 9. Let z be the solution of problem III_3 and v the solution of the corresponding problem III_2 (i.e. that which has the same coefficients and the same right-hand sides). Then for $\|\tau\|_0 < \tau_0$ we have the estimate

$$\|z(x, t_{j+1}) - v(x, t_{j+1})\| + \\ + \sqrt{\tau}\|z_{\bar{x}}(x, t_{j+1}) - v(x, t_{j+1})\| \leqslant M\|\sqrt{\tau}\, v_{\bar{x}}(x, t_{j+1})\|, \tag{80}$$

where

$$\|v_{\bar{x}}(x, t_{j+1})\| = \sum_{\alpha=1}^{p} \|v_{\bar{x}_\alpha}(x, t_{j+\alpha/p})\|.$$

For the function $w = z - v$ is the solution of problem III_3° with the right-hand side

$$\widetilde{\psi}_\alpha = -\left(b_\alpha^+ v_{\hat{x}_\alpha} + b_\alpha^- v_{\check{x}_\alpha}\right) + \delta_{\alpha,p}\sum_{k=1}^{p}\left(b_k^+ v_{\hat{x}_k} + b_k^- v_{\check{x}_k}\right), \qquad \delta_{\alpha,p} = \begin{cases} 1, & \alpha = p, \\ 0, & \alpha \neq p, \end{cases}$$

which satisfies condition (70) and this enables us to use the estimate (61).

Lemma 10. Let z be the solution of $\mathrm{III}_{2'}$ and v the solution of the corresponding problem $\mathrm{III}_{2'}$ for $g_{\alpha 1}^{\pm} = 0$. If the net ω_τ is quasi-uniform $(|\tau_{\bar{t}}| \leqslant m^* \tau)$, then for $\|\tau\|_0 \leqslant \tau_0$ we have the estimate

$$\|z - v\|^{j+1} + \sqrt{\tau_{j+1}}\|z_{\bar{x}} - v_{\bar{x}}\|^{j+1} \leqslant M\|\sqrt{\tau_{j+1}}\, v_{\bar{x}}(x, t_{j+1})\|. \tag{81}$$

For the difference $w = z - v$ we obtain problem $\mathrm{III}_{2'}$ with right-hand side

$$\widetilde{\psi}_\alpha = \left(b_{\alpha-1}^+ v_{\hat{x}_{\alpha-1}} - b_\alpha^+ v_{\hat{x}_\alpha}\right) + \left(b_{\alpha-1}^- v_{\check{x}_{\alpha-1}} - b_\alpha^- v_{\hat{x}_\alpha}\right), \qquad \alpha = 2, \ldots, p,$$

$$\widetilde{\psi}_1 = \left[\left(b_p^+ v_{\hat{x}_p}\right)^j - b_1^+ v_{\hat{x}_1}\right] + \left[\left(b_p^- v_{\check{x}_p}\right)^j - b_1^- v_{\check{x}_1}\right], \qquad \alpha = 1,$$

which satisfies the condition

$$\sum_{\alpha=1}^{p} \widetilde{\psi}_\alpha = -\tau q_{\bar{t}}, \qquad q = b_p^+ v_{\hat{x}_p} + b_p^- v_{\check{x}_p}, \qquad q_{\bar{t}} = (q^{j+1} - q^j)/\tau. \tag{82}$$

The last term on the right-hand side of inequality (46) for the function w will have the form

$$2\sum_{\alpha=1}^{p} (\widetilde{\psi}_\alpha, w^{j+\alpha/p})\,\tau = -2\sum_{\alpha=1}^{p-1} (\widetilde{\psi}_\alpha, w^{j+1} - w^{j+\alpha/p})\,\tau - 2\tau\,(q_{\bar{t}}, (\tau w)^{j+1}). \tag{83}$$

Reasoning as in Para. 4 we find

$$2\sum_{\alpha=1}^{p-1}(\widetilde{\psi}_\alpha, w^{j+1}-w^{j+\alpha/p})\,\tau \leqslant \tfrac{1}{2}\tau P_1 + M\tau^2\sum_{\alpha=1}^{p-1}\|\widetilde{\psi}_\alpha\|^2.$$

Let us transform the last term on the right-hand side of (83):

$$-2\tau\,(q_{\bar{t}}, (\tau w)^{j+1}) = -2\tau\,(\tau q, w)_{\bar{t}} + 2\tau\check{\tau}\,(q^j, w_{\bar{t}}) + 2\tau\tau_{\bar{t}}\,(w^{j+1}, q^j) \leqslant$$
$$\leqslant -2\tau\,(\tau q, w)_{\bar{t}} + \tfrac{1}{2}\tau P_1 + M\tau\|w^{j+1}\|^2 + M\tau\|\sqrt{\check{\tau}}\, q^j\|^2 \qquad (\tau = \tau_{j+1}, \check{\tau} = \tau_j).$$

As a result, instead of (46) we obtain

$$(\rho, w^2)^{j+1} + \tau I_1 \leqslant (1 + M\tau)(\rho, w^2)^j + M\tau\,\{\|\sqrt{\check{\tau}}\, q^j\|^2 + \sum_{\alpha=1}^{p}\|\sqrt{\tau}\,\widetilde{\psi}_\alpha\|^2\} - 2\tau\,(\tau q, w)_{\bar{t}}, \tag{84}$$

$$I_1^{(\alpha)} = (a_\alpha, w^2_{\bar{x}_\alpha}]_\alpha, \qquad I_1 = \sum_{\alpha=1}^{p} I_1^{(\alpha)}.$$

Summing this inequality over all $j' = 0, 1, 2, \ldots, j$ and then using Lemma $4a$ [14] we obtain

$$(\rho, w^2)^{j+1} + \tau I_1^{j+1} \leqslant M\overline{R(t_{j+1})} + M'\,|(\tau q, w)^{j+1}|, \tag{85}$$

where R is the expression in curly brackets in (84). Estimating $M'\,|(\tau q, w)^{j+1}| \leqslant 0.5\,(\rho, w^2)^{j+1} + M\tau^2\|q^{j+1}\|^2$, we arrive at (81).

If we know an estimate for $\|v\| + \sqrt{\tau}\,\|v_{\bar{x}}\|$, then Lemma 10 enables us to find an estimate for $\|z\| + \sqrt{\tau}\,\|z_{\bar{x}}\|$.

Lemma 11. Let z be the solution of problem III_1 with the right-hand side $\Psi_\alpha = 0$. Then for sufficiently small $\|\tau\|_0 < \tau_0$ on the arbitrary net $\Omega^{(1)}$ we have the inequality

$$\sqrt{\tau_{j+1}}\,\|z_{\bar{x}_\alpha}(x, t_{j+\alpha/p})\|_{2_\alpha,0} \leqslant M\|\sqrt{\tau}\,\nu\|^0_{0,s}\, h^{-1/2}_{*,\gamma}, \tag{86}$$

where

$$h_{*,\gamma} = \min_{x\in\gamma_\alpha,\ \alpha=1,2,\ldots,p}(\hbar_\alpha, h_{\alpha+}).$$

From Theorem 1 of [10] we have $\|z(x, t_{j+1})\|_0 \leqslant M\|\nu\|_{0,s}$. Let us write a first order inequality for the chain Π_α assuming that the boundary conditions are not homogeneous $(z = \beta_\alpha^{\pm} z^{(\mp 1_\alpha)} + \nu_\alpha^{\pm}, \quad x \in \gamma_\alpha^{\pm})$:

$$(\rho, z^2)^{*}_{u_\alpha} + \tau\,(\rho, z^2_{\bar{t}_\alpha})^{*}_{u_\alpha} + \tau\,(a_\alpha, z^2_{\bar{x}_\alpha}]_{u_\alpha} \leqslant (1 + M\tau)(\check{\rho}, \check{z}^2)^2_{u_\alpha} + A_\alpha,$$

where

$$A_\alpha = 2\tau a_\alpha z_{\bar{x}_\alpha} z \Big|_{x=x^{(+\alpha)}} - 2\tau a_\alpha^{(+1_\alpha)} z_{x_\alpha} z \Big|_{x=x^{(-\alpha)}}.$$

In this we put z from the boundary conditions $z = h_{\alpha+}\varkappa_\alpha^- z_{x_\alpha} + \nu_\alpha^-$, $z = h_\alpha \varkappa_\alpha^+ z_{\bar{x}_\alpha} + \nu_\alpha^+$. Then

$$A_\alpha = -2\tau\left\{a_\alpha^{(+1_\alpha)} h_{\alpha+} z_{x_\alpha}^2 \varkappa_\alpha^- + a_\alpha h_\alpha z_{\bar{x}}^2 \varkappa_\alpha^+\right\} + 2\tau\left(a_\alpha z_{\bar{x}_\alpha} \nu_\alpha^+ - a_\alpha^{(+1_\alpha)} z_{x_\alpha} \nu_\alpha^-\right).$$

Since $2\tau a_\alpha z_{\bar{x}_\alpha} \nu_\alpha^+ < \frac{1}{2}\tau a_\alpha z_{\bar{x}_\alpha}^2 h_\alpha + a_\alpha (\nu_\alpha^+ h_\alpha^{-1/2})^2 \tau\cdot 2$, we obtain

$$\sqrt{\tau}(a_\alpha, z_{\bar{x}_\alpha}^2]^{1/2} < (1 + M\tau)\|\check{z}\|_{0,u_\alpha} + M\sqrt{\tau}\,[|\nu_\alpha^+|h_\alpha^{-1/2} + |\nu_\alpha^-|h_{\alpha+}]^{+1/2}. \tag{87}$$

If, for example, $\nu_\alpha^- = O(h_{\alpha+}^2)$, $\nu_\alpha^+ = O(h_\alpha^2)$, then it follows from (87) that $\sqrt{\tau}\|z_{\bar{x}_\alpha}\|_{2_\alpha,0} \leqslant M\|h\|_{0,\gamma}^2 + M\tau$.

Noting that

$$\|z_{\bar{x}}\| \leqslant M\sum_{\alpha=1}^{p}\|z_{\bar{x}_\alpha}\|_{2_\alpha,0},$$

we can see that the following theorem is true.

Theorem 5. Let z be the solution of problem $III_{1'}$ for $\Psi_\alpha = 0$ with the boundary conditions $z = \beta_\alpha^\pm z^{(\mp 1_\alpha)} + \nu_\alpha^\pm$, $x \in \gamma_\alpha^\pm$. If ω_τ is a quasi-uniform net then

$$\|z\|^{j+1} + \sqrt{\tau_{j+1}}\|z_{\bar{x}}\|^{j+1} \leqslant M\|\nu\|_{0,*} + M\sqrt{\|\tau\|_0}\sum_{\alpha=1}^{p}\max_{\gamma_\alpha^+ \times \omega_\tau}|\nu_\alpha^+|h_\alpha^{-1/2} + \max_{\gamma_\alpha^+ \times \omega_\tau}|\nu_\alpha^-|h_{\alpha+}^{-1/2}]. \tag{88}$$

3. On the convergence and accuracy on non-uniform nets

1. INTRODUCTION

In order to clarify the question of the convergence and accuracy of the schemes II_k (k = 1, 2, 3, 1′, 2′) examined in Section 1 let us use the *a priori* estimates for the solution z of problem III_k which is the error of the solution y of the problem II_k ($z = y - u$), obtained in Section 2. These estimates take the structure of the approximation error into account, and for all the schemes it has the form

$$\Psi_\alpha = \mathring{\psi}_\alpha + \psi_\alpha, \qquad \sum_{\alpha=1}^{p}\mathring{\psi}_\alpha = 0, \qquad \mathring{\psi}_\alpha = O(1), \tag{1}$$

$$\psi_\alpha = \sum_{\beta=1}^{p} [(\mu_{\alpha\beta})_{\hat{x}_\beta} + \psi^*_{\alpha\beta}], \tag{2}$$

$$\mu_{\alpha\beta} = O(h_\beta^2), \qquad \psi^*_{\alpha\beta} = O(\hbar_\beta^2) + O(\tau). \tag{3}$$

On the uniform net $\omega_h^{(1)}$ we must put $\mu_{\alpha\beta} = 0$, $\psi^*_{\alpha\beta} = \delta_{\alpha\beta}\psi^*_{\alpha\alpha}$, $\psi^*_{\alpha\alpha} = O(\hbar_\alpha^2) + O(\tau)$ where $\psi^*_{\alpha\alpha} = O(\hbar_\alpha) + O(\tau)$ in the case of the net $\omega_h^{(2)}$ at nodes adjacent to the boundary γ_α. Nevertheless, as we shall show below, this does not reduce the order of accuracy of the schemes.

Let us formulate the conditions laid down on the solution $u = u(x, t)$ of the problem I_k and on the coefficients of the differential equation (Section 1, 1_k, $k = 1, 2$).

We shall take it for granted everywhere below that the following condition is fulfilled:

in $\bar{Q}_T$ $c(x, t)$ satisfies the Lipschitz condition with respect to t, so that

$$|c_{\bar{t}}| \leqslant c_5, \qquad c_5 = \text{const.} > 0. \tag{a}$$

All the *a priori* estimates in Section 2 were obtained for this condition, i.e. for $|\rho_{\bar{t}}| \leqslant c_5$.

Condition A. The solution $u = u(x, t)$ of the problems I_k ($k = 1, 2, 3$) and the functions $c(x, t)$, $k_\alpha(x, t, u)$, $f(x, t, u, q_1, \ldots, q_p)$ possess all the derivatives which are sufficient for formulae (1)-(3) for Ψ_α to hold at any point of the net $\Omega^{(1)}$ or $\Omega^{(2)}$.

Condition B. The functions $\overset{\circ}{\psi}_\alpha/c(x, t)$ satisfy in $\bar{Q}_T$ the Lipschitz conditions with respect to the variables x_β where $\beta \leqslant \alpha$ for $\alpha \leqslant [p/2]$ or $\beta > \alpha$ for $\alpha > [p/2]$, $\alpha, \beta = 1, 2, \ldots, p$.

Condition C. The expression $\Lambda_\beta(\overset{\circ}{\psi}_\alpha/c(x, t))$ is uniformly bounded on any net $\Omega^{(1)}$ or $\Omega^{(2)}$ for all $\beta \leqslant \alpha$ for $\alpha \leqslant [p/2]$ or $\beta > \alpha$ for $\alpha > [p/2]$, $\alpha = 1, 2, \ldots, p$.

Let us illustrate condition A using the example of problem I_0. For condition A to be satisfied it is sufficient, for example, that $\partial u/\partial t$, u, k_α, f, r_α, $\partial^3 u/\partial x_\alpha^3$, $\partial c/\partial x_\alpha$, $\partial r_\alpha/\partial x_\alpha$, $\partial^2 k_\alpha/\partial x_\alpha^2$, $\partial f/\partial x_\alpha$ shall satisfy the Lipschitz conditions for x_α, $\alpha = 1, 2, \ldots, p$ and $c(x, t)$, $\partial u/\partial t$ the Lipschitz condition for t.

Since we have agreed to take all the coefficients $\rho = \rho(x, t_{j+1})$,

$a_\alpha = a_\alpha(x, t_{j+1})$, $b_\alpha^\pm = b_\alpha(x, t_{j+1})$, $\varphi_\alpha = \varphi_\alpha(x, t_{j+1})$ at time $t = t_{j+1}$, it is not necessary for the coefficients $k_\alpha(x, t)$, $r_\alpha(x, t)$, $q(x, t)$, $f(x, t)$ to satisfy the Lipschitz condition with respect to t (in [10] this was necessary when the principal part $\overset{\circ}{\psi}_\alpha$ of the approximation error Ψ_α was taken separately).

The results of the work [10] are obviously valid for problem II_0 if conditions A and C are satisfied.

With the additional requirement that $\partial u/\partial t$ shall satisfy the Lipschitz conditions with respect to x_α conditions A are equivalent to the conditions under which every one-dimensional scheme Π_α has the maximum order of approximation $O(h_\alpha^2) + O(\tau)$ on a uniform net.

We shall conduct our next argument for the scheme II_1, only briefly indicating the results for other schemes.

2. ON THE CONVERGENCE OF THE SCHEME II_1

It follows from [10] that schemes II_0 and II_1 have accuracy $O(h^2) + O(\tau)$ on the uniform net $\Omega^{(1)}$ if conditions A and C are satisfied. The method of [10] did not allow us to examine the question of the convergence when weaker conditions were laid on u, c, k_α, r_α, f even in the case of the uniform net $\Omega^{(1)}$. This can now be done, using the *a priori* estimates of Section 2.

We shall need conditions $A^{(0)}$ and $A^{(\sigma)}$:

$A^{(0)}$. The functions k_α, c, f, $\partial k_\alpha/\partial x_\alpha$, $\partial r_\alpha/\partial u$, r_α, $\partial u/\partial x_\alpha$, $\partial^2 u/\partial x_\alpha^2$, $\partial u/\partial t$, $\alpha = 1, 2, \ldots, p$ are uniformly continuous in $\bar{Q}_T$.

$A^{(\sigma)}$. The functions listed in $A^{(0)}$ satisfy in $\bar{Q}_T$ the Hölder conditions of order $\sigma_1 > 0$ with respect to x_α, $\alpha = 1, \ldots, p$ and $\partial u/\partial t$ in addition satisfies the Hölder condition of order $\sigma_2 > 0$ with respect to t.

It is not difficult to see that

$$\Psi_\alpha = \overset{\circ}{\psi}_\alpha + \psi_\alpha, \qquad \sum_{\alpha=1}^{p} \overset{\circ}{\psi}_\alpha = 0, \tag{4}$$

$$\psi_\alpha = \rho(\hbar) + \rho(\tau), \quad \text{if the conditions } A^{(0)} \text{ are fulfilled} \tag{5}$$

($\rho(\varepsilon) \Rightarrow 0$ when $\varepsilon \to 0$),

$$\psi_\alpha = O(\hbar^{\sigma_1}) + O(\tau^{\sigma_2}), \quad \text{if the conditions } A^{(\sigma)}, \text{ are fultilled} \quad \hbar = \sum_{\alpha=1}^{p} \hbar_\alpha \tag{5'}$$

on arbitrary nets $\Omega^{(1)}$ and $\Omega^{(2)}$.

Theorem 6. If conditions $A^{(0)}$ are satisfied, then the scheme II_1 converges in the mean on an arbitrary sequence of nets $\Omega^{(1)}$ and $\Omega^{(2)}$:

$$\| y - u \| = \rho_1 (\| h \|_0) + \rho_2 (\| \tau \|_0) \text{ for } \| \tau \|_0 \leqslant \tau_0, \qquad (6)$$

where $\rho_k(\xi) \to 0$ as $\xi \to 0$, $k = 1, 2$.

Proof. For the error $z = y - u$ of the scheme II_1 we obtain problem III_1, where Ψ_α is given by formulae (4) and (5). Let us put z in the form of the sum $z = \bar{z} + v$, where $\bar{z}$ is the solution of this problem III_1 for $\Psi_\alpha = 0$, and v is the solution of the same problem with homogeneous boundary conditions ($v_\alpha^\pm = 0$); $z \equiv 0$ on the net $\Omega^{(2)}$. Due to conditions $A^{(0)}$ we have $v_\alpha^\pm = O(h_\alpha)$ and Theorem 1 of [10] gives $\|\bar{z}\|_0 = O(\|h\|^2_{0,\gamma})$. To estimate v we make use of Theorem 1. Remembering that $\|\Psi\| = \rho(\hbar) + O(\sqrt{\tau}) + \rho(\tau)$, we find $\|v\| = \rho_1(\|\hbar\|)_0 + \rho_2(\|\tau\|_0)$. For the problems II_k, $k = 2, 3, 1', 2'$ there is an analogous theorem.

In order to prove uniform convergence conditions $A^{(0)}$ are insufficient, due to the factor $\ln^\delta (1/H_*)$ in the estimate (2.73). Due to this factor in the case of a non-uniform net ω_h we must introduce an additional restriction on the magnitude of H_*:

$$H_* > \exp(-c/\|H\|_0^\varepsilon), \quad H_* = \min_{x \in \omega_h} H, \quad H = \prod_{\alpha=1}^{p} \hbar_\alpha, \qquad (7)$$

where ε is a positive number as small as we please, and c is an arbitrary positive constant. It follows from condition (7) that

$$\ln \frac{1}{H_*} < \frac{c}{\|H\|_0^\varepsilon}.$$

It is not difficult to see that this requirement is not very onerous and is a weak restriction on the arbitrary selection of the nets ω_h. When formulating our theorems concerning uniform convergence below we shall take it for granted that condition (7) is satisfied.

In proving uniform convergence we shall use Theorem 3. Estimate (73) contains the norms $\|\psi_\alpha\|_{4_\alpha,0} = \| \, \|\psi_\alpha\|_{4_\alpha,1} \|_0$, where $\|\psi_\alpha\|_{4_\alpha,1}$ is the norm along the chain Π_α, defined by formula (2.33). Let us introduce the concept of the mean square step along a given direction α. We take some chain Π_α of length l_α. It is obvious that l_α is a net function on ω_h which takes constant values along each of the chains Π_α. Let Π_α^* be the chain of direction α having the greatest length $l_*^* = \|l_\alpha\|_0$. We define the mean square step in the direction α to be the quantity

$$\|h_\alpha\|^*_{2_\alpha} = \frac{1}{l^*_\alpha}(1, h^2_\alpha]^{1/2}_{\gamma_\alpha}. \tag{8}$$

We recall that

$$\|h_\alpha\|_{0,\gamma_\alpha} = \max\,(\max_{x\in\gamma^-_\alpha} h_{\alpha+}, \max_{x\in\gamma^+_\alpha} h_\alpha),$$

$$\|h\|_{0,\gamma} = \sum_{\alpha=1}^{p} \|h_\alpha\|_{0,\gamma_\alpha}.$$

Theorem 7. If conditions $A^{(\sigma)}$ are satisfied, scheme II_1, for $\rho(x,t) = c(\widetilde{x,t})$ converges uniformly on the non-uniform nets $\Omega^{(1)}$ and $\Omega^{(2)}$:

$$\|y-u\|_0 \leqslant M(\|h^{\sigma_1}\|^* + \varepsilon_k\|h\|^2_{0,\gamma} + \|\tau\|_0^{\sigma'_2})\ln^\delta \frac{1}{H_*}$$
$$\text{for } \|\tau\|_0 \leqslant \tau_0,\ H_* \leqslant H_0(\delta), \tag{9}$$

where $\sigma'_2 = \min\,(0.5, \sigma_2) \leqslant 0.5$, $\delta > 1$ is an arbitrary number

$$\|h\|^* = \sum_{\alpha=1}^{p} \|h_\alpha\|^*_{2_\alpha}, \qquad \varepsilon_k = \delta_{k,1} \text{ on the net } \Omega^{(k)}, \quad k=1,2. \tag{10}$$

The proof is also based on the representation of $z = y - u$ in the form of the sum $\bar{z} = \bar{z} + v$; for $\bar{z}$ we have the estimate $\|\bar{z}\|_0 = O(\|h\|^2_{0,\gamma})$ We use Theorem 3 to estimate v, and take into account the fact that $\mu_{\alpha\beta} \equiv 0$ and $\|\psi_\alpha\|_{4_{\alpha,1}} = O(\|h^{\sigma_1}_\alpha\|^*_{2_\alpha})$.

3. ON THE ORDER OF ACCURACY ON A NON-UNIFORM NET

Theorem 8. If conditions A and B are satisfied, then on an arbitrary sequence of non-uniform nets $\Omega^{(1)}$ and $\Omega^{(2)}$ scheme II_1 has first order accuracy with respect to τ and second order accuracy with respect to h, so that when $\|\tau\|_0 < \tau_0$ we have the estimates

$$\|y-u\| \leqslant M(\|h^2\|_2 + \varepsilon_k\|h^2\|_{0,\gamma} + \|\tau\|_0), \tag{11}$$

$$\|y-u\|_0 \leqslant M(\|h\|^2_0 + \|\tau\|_0)\ln^\delta \frac{1}{H_*} \text{ for } H_* \leqslant H_0(\delta), \tag{12}$$

$$\varepsilon_k = \delta_{k,1} = \begin{cases} 1, & k=1 \\ 0, & k=2 \end{cases} \text{ on the net } \Omega^{(k)}.$$

Proof. Let us put $z = y - u$ in the form of the sum $z = \eta + \bar{z} + v$ where η is the function defined in [10], Section 2, Para. 2 $(\rho\eta_{\bar{t}_\alpha} = \mathring{\psi}_\alpha)$, $\bar{z}$ is the solution of problem III_1 with the right-hand side $\Psi_\alpha = 0$ and non-homogeneous boundary conditions $\bar{z} = \beta_\alpha^{\pm}\bar{z}^{(\mp 1_\alpha)} + \tilde{v}_\alpha^{\pm}$ when $x \in \gamma_\alpha^{\pm}$, where $\tilde{v}_\alpha^{\pm} = \beta_\alpha v_\alpha^{\pm} - \eta + \beta_\alpha^{\pm}\eta^{(\mp 1_\alpha)}$, when $\beta_\alpha^{\pm} = 0$ on $\Omega^{(2)}$ and v is the solution of problem III_1 with homogeneous boundary conditions and the right-hand side $\Psi_\alpha = \psi_\alpha + \Lambda_\alpha\eta + Q_\alpha^{(1)}(\eta)$. By analogy with [10] we find $\|\eta\|_0 = O(\tau)$. Theorem 1 of [10] gives $\|\bar{z}\|_0 = O(\|\tau\|_0 + \|h^2\|_{0,\gamma})$ on $\Omega^{(1)}$ and $\|\bar{z}\|_0 = O(\|\tau\|_0)$ on $\Omega^{(2)}$. To estimate v we use Theorems 2 and 3, remembering that

$$\psi_\alpha = \sum_{\beta=1}^{p} [(\mu_{\alpha\beta})_{\hat{x}_\beta} + \psi^{*}_{\alpha\beta}].$$

Let us first calculate $\|\Psi\|$ using formulae (57) and (60), putting

$$\mu_\alpha = \sum_{\beta=1}^{p} \mu_{\beta\alpha} = \sum_{\beta=1}^{p} \mathring{\mu}_{\beta\alpha} h_\alpha^2.$$

The calculation gives

$$\|\Lambda_\alpha\eta\|_{4_\alpha} = O(\|\eta_{\bar{x}_\alpha}\|_{2_\alpha}) = O(\tau), \qquad \|Q_\alpha^{(1)}(\eta)\|_{4_\alpha} = O(\tau), \quad \|\mu_\alpha\|_{2_\alpha} = O(\|h_\alpha^2\|_{2_\alpha}).$$

Combining the estimates for η, $\bar{z}$ and v we obtain (11). Let us now return to Theorem 3 and Lemma 8.

Noting that $\|\Psi\| = O(\|h\|_0^2) + O(\tau)$, we obtain (12).

Theorem 9. If conditions A are satisfied, then the solution of problem II_1 on $\Omega^{(1)}$ and $\Omega^{(2)}$ for $\|\tau\|_0 \leqslant \tau_0$ satisfies the inequalities

$$\|y - u\| \leqslant M\,(\|h^2\| + \varepsilon_k\|h^2\|_{0,\gamma} + \|\tau\|_2^{1/2}), \tag{13}$$

$$\|y - u\|_0 \leqslant M\,(\|h^2\|_0 + \|\tau\|_0^{1/2})\ln^5\frac{1}{H_*}, \qquad H_* \leqslant H_0(\delta). \tag{14}$$

These estimates follow at once from Theorems 2, 3 and Lemma 8.

Estimate (13) is valid for the schemes II_2, II_3, $\mathrm{II}_{1'}$, $\mathrm{II}_{2'}$. A comparison of Theorems 8 and 9 shows that by weakening the restrictions on the solution and on the coefficients of the differential equation we obtain the cruder estimate (13) instead of (11).

Assuming that conditions A and B are satisfied and using Theorem 5 we can obtain the following estimate of the rate of convergence (in the mean) for the scheme II_1, on the net $\Omega^{(1)}_{**}$ (ω_τ is a quasi-uniform net):

$$\|y-u\| \leqslant M\left(\|h^2\|+\|\tau\|_0+\|\tau\|_0^{3/2}/h_{*,\gamma}^{1/2}\right),\ \|\tau\|_0 \leqslant \tau_0,$$

where

$$h_{*,\gamma}=\min_{x\in\gamma_\alpha,\alpha=1,2,\ldots,p} h_\alpha.$$

In the case of a system of N parabolic equations, by analogy with [14], Section 4, Paragraph 5, we can write down an absolutely stable local one-dimensional scheme. To solve the resulting difference equations with respect to the vector $y^{j+\alpha/p}$ the $2N$-times application of one-dimensional successive substitution is required. This scheme converges in the mean at a rate $O(h^2) + O\ (\sqrt{\tau})$ (cf. Theorem 9).

4. Appendix

1. THE THIRD BOUNDARY PROBLEM FOR A PARALLELEPIPED

Let $\overline{G}=\{0\leqslant x_\alpha\leqslant l_\alpha,\ \alpha=1,\ldots,p\}$ be a parallelepiped on the boundary of which conditions of the third kind are given $\left(k_\alpha\dfrac{\partial u}{\partial x_\alpha}=\sigma_\alpha^- u+u_{1\alpha}^-\right.$ when $x_\alpha=0,\ -k_\alpha\dfrac{\partial u}{\partial x_\alpha}=\sigma_\alpha^+ u+u_{1\alpha}^+$ when $\left.x_\alpha=l_\alpha\right)$. The corresponding problems I_1' and II_1' where considered in [10], where we took the simplest difference boundary conditions of the first order of approximation for II_1':

$$a_\alpha^{(+1_\alpha)}y_{x_\alpha}=\sigma_\alpha^- y+u_{1\alpha}^- \text{ for } x_\alpha=0,\quad -a_\alpha y_{\bar{x}_\alpha}=\sigma_\alpha^+ y+u_{1\alpha}^+ \text{ for } x_\alpha=l_\alpha.$$

For the error $z = y - u$ of the scheme II_1' on the non-uniform net* $\overline{\Omega}_0$ we obtain problem III_1' with the boundary conditions

$$a_\alpha^{(+1_\alpha)}z_{x_\alpha}=\sigma_\alpha^- z+\nu_\alpha^- \text{ for } x_\alpha=0,\quad -a_\alpha z_{\bar{x}_\alpha}=\sigma_\alpha^+ z+\nu_\alpha^+ \text{ for } x_\alpha=l_\alpha,\tag{1}$$

where $\nu_\alpha^-=O(h_{\alpha+}),\ \nu_\alpha^+=O(h_\alpha)$. By analogy with Section 2 of [10] we find the *a priori* estimate

$$\|z(x,t)\|\leqslant M\,[\,\|\overline{\psi(x,t)}\|+\|\overline{\nu(x,t)}\|_{0,\gamma})\ \text{for}\ \|\tau\|_0\leqslant\tau_0,\tag{2}$$

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* If the region G is a parallelepiped, we shall denote the net $\overline{\Omega}^1$ by $\overline{\Omega}_0$ below.

$$\|v\|_{0,\gamma}=\sum_{\alpha=1}^{p}\left(\|v_\alpha^-\|_{0,\gamma_\alpha^-}+\|v_\alpha^+\|_{0,\gamma_\alpha^+}\right),\|v_\alpha^\pm\|_{0,\gamma_\alpha^\pm}=\max_{x\in\gamma_{\alpha^\pm}}|v_\alpha^\pm|\left(\|\tau\|_0=\max_{0<t_{j'}\leqslant t}\tau_{j'}\right),$$

which holds if $\sigma_\alpha^\pm\geqslant c_1>0$. Here $\overline{\|\Psi(x,t)\|}$ is the expression (2.40). The following estimate of the order of accuracy is true (for conditions A and B of Section 3) for the scheme II_1':

$$\|y-u\|\leqslant M\,(\|\tau\|_0+\|\hbar^2\|+\|h\|_{0,\gamma}), \tag{3}$$

i.e. generally speaking the scheme II_1' has first order accuracy with respect to h. Concentrating the net near the boundary, or, more exactly, choosing $\|h\|_{0,\gamma}=O(\|\hbar^2\|)$, we can increase the accuracy on a non-uniform net to second order $(\|y-u\|=O(\|\hbar^2\|)+O(\|\tau\|_0))$.

It is not difficult to obtain uniform estimates also, by analogy with Section 2, Para. 5 and Section 3, Para. 3. The results (2) and (3) are also true of the scheme II_2.

2. DISCONTINUOUS COEFFICIENTS

The *a priori* estimates obtained in Section 2 enable us to examine in detail the question of the accuracy of the schemes II_k in the class of coefficients of the differential equation (1.1_k) with discontinuities of the first kind on a finite number of hyperplanes $x_\alpha = \xi_\alpha$ = const. parallel to the coordinate hyperplanes. We can formulate all the results by analogy with the one-dimensional case [14]. If the net ω_h is arbitrary, then the estimate for $z = y - u$ will contain first powers of the steps h_α at the points adjacent to a discontinuity (cf. [13]). Concentrating the net near the discontinuity $x_\alpha = \xi_\alpha$, i.e. choosing $h_\alpha = O(\|\hbar^2\|)$ at these points we obtain second order accuracy with respect to h on such a net. The nets $\omega_h(k)$ introduced in [14] are more economical; they are selected so that the nodes of the net lie on the hyperplanes $x_\alpha = \xi_\alpha$. In this case, as we showed in Section 1, the expression (2.5) for ψ_α is valid at all points of the net $\omega_h(k)$ and hence our schemes have the same order of accuracy as in the class of continuous coefficients.

3. ELLIPTIC EQUATIONS

The methods of [14]-[17] and of this paper are also applicable to the study of the convergence and accuracy of homogeneous difference schemes for multi-dimensional equations of elliptic type.

Suppose that in the region $\bar{G}$ we are given the boundary problem

$$Lu - q(x)u = -f(x), \qquad u|_\Gamma = u_1(x), \qquad Lu = \sum_{\alpha=1}^{p} L_\alpha u, \qquad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\right), \tag{4}$$

$$k_\alpha \geqslant c_1 > 0, \qquad q(x) \geqslant 0.$$

The corresponding multi-dimensional homogeneous difference scheme on a non-uniform net $\omega_h^{(2)}$ has the form

$$\Lambda y = \sum_{\alpha=1}^{p} \Lambda_\alpha y = d(x)y - \varphi(x) \text{ for } x \in \omega_h^{(2)}, \quad u|_\gamma = u_1(x); \qquad \Lambda_\alpha y = (a_\alpha(x) y_{\bar{x}_\alpha})_{\hat{x}_\alpha}. \tag{5}$$

The coefficients a_α, d and φ are calculated from the formulae of Section 1, Para. 3.

For $z = y - u$ we obtain the problem

$$\Lambda z - dz = \Psi, \quad x \in \omega_h^2; \quad z|_\gamma = 0; \quad \Psi = \sum_{\alpha=1}^{p} [(\mu_\alpha)_{\hat{x}_\alpha} + \psi_\alpha^*], \quad \mu_\alpha = O(h_\alpha^2), \quad \psi_\alpha^* = O(h_\alpha^2). \tag{6}$$

A priori estimates for z can be constructed, as in Section 2, using n-th rank energy inequalities. We majorise $2^n(\Psi, z^{\sigma_n})$ for $n > 1$ thus:

$$2^n(\Psi, z^{\sigma_n}) < I_n + \left(M \times 2^n \sum_{\alpha=1}^{p} \|\Psi_\alpha\|_{3_{\alpha,0}}\right)^{2^n}, \qquad I_n = \sum_{\alpha=1}^{p} \times I_n^{(\alpha)}, \quad \Psi = \sum_{\alpha=1}^{p} \Psi_\alpha.$$

For $n = 1$ we have the estimate

$$\|z\| < M \sum_{\alpha=1}^{p} (\|\mu_\alpha\|_{3_\alpha} + \|\psi_\alpha^*\|_{3_\alpha}) \quad \|\psi_\alpha^*\|_{3_\alpha} = \|\eta_\alpha\|_{3_\alpha}, \quad (\eta_\alpha)_{\hat{x}_\alpha} = \psi_\alpha^*, \quad \eta_\alpha^{(+1)_\alpha} = 0, \qquad x \in \gamma_\alpha^-. \tag{7}$$

On the arbitrary non-uniform net $\omega_h^{(2)}$ the difference scheme (5) has second order accuracy:

$$\|y - u\| \leqslant M\|h^2\| \text{ and } \|y - u\|_0 \leqslant M\|h\|_0^2 \ln^\delta \frac{1}{H_*}, \quad H_* \leqslant H_0(\delta). \tag{8}$$

The net $\omega_h^{(2)}$ is obviously more convenient than the net $\omega_h^{(1)}$ since the procedure of extrapolation of the boundary conditions from Γ on γ in this case is less convenient than for the local one-dimensional method.

It is clear from what we have said that these schemes converge in the class of coefficients having discontinuities of the type described in Para. 2.

In [15], [19], [20] a method of putting Ψ in "divergent" form and then applying the first rank energy identity was used. More refined estimates, based on the use of Green's function, were obtained in [18].

4. EXPLICIT SCHEMES

The advantage of local one-dimensional schemes over multi-dimensional schemes is easily appreciated even in the case of purely explicit schemes. Let us illustrate this remark, taking equation (1.1_0) with constant coefficients and putting, for simplicity, $r_\alpha = q = f = 0$, $h_\alpha = c = 1$, taking the net ω_h to be uniform and "square" ($h_\alpha = h = \text{const.}$).

Let us write down an explicit multi-dimensional scheme and the corresponding local one-dimensional scheme (cf. [3]):

$$y_{\bar{t}} = (y^{j+1} - y^j)/\tau = \Lambda y^j, \qquad \Lambda y = \sum_{\alpha=1}^{p} \Lambda_\alpha y, \qquad \Lambda_\alpha y = y_{\bar{x}_\alpha x_\alpha}, \tag{9}$$

$$y_{\bar{t}_\alpha} = \Lambda_\alpha \check{y}, \quad \alpha = 1, 2, \ldots, p; \qquad y = y^{j+\alpha/p}, \qquad \check{y} = y^{j+(\alpha-1)/p}. \tag{10}$$

The stability conditions for them have the form

$$\frac{\tau}{h^2} \leqslant \frac{1}{2p} \quad \text{for (9)}, \qquad \frac{\tau}{h^2} \leqslant \frac{1}{2} \quad \text{for (10)}.$$

Thus, the local one-dimensional method allows us to increase the step with respect to time p times without increasing the amount of calculation required.

5. ON SPLITTING METHODS

In [3]-[5], [7]-[9], [11] the question of the accuracy of economical schemes, which the authors call splitting methods, reduces to the investigation of a multi-dimensional difference equation connecting the values y^j and y^{j+1} on integral steps, and this greatly restricts both the class of equations and the class of regions G for which these methods are applicable. While this method is suitable for the schemes of [7]-[9] in our opinion a more natural method for the schemes of [3]-[5] is that described in [10] and [12]. The fact is that in the simplest cases considered in [3]-[5] the algorithms of [3] and [10] are the same. In this connection it should be noted that the works [3]-[5] gave the stimulus to the author for the work of [10] and [12].

In [3], [11] the following six-point schemes were used:

$$y_{\bar{t}\alpha} = \Lambda_\alpha (\sigma y^{j+\alpha/p} + (1-\sigma) y^{j+(\alpha-1)/p}), \qquad 0 \leqslant \sigma \leqslant 1.$$

In [12], for $p = 2$, $\sigma = 0.5$ it is shown that these schemes converge in the mean at the rate $O(\| h^2 \|) + O(\| \tau \|_2)$ on an arbitrary sequence of non-uniform nets $\omega_h^{(2)}$. When $p > 2$ this estimate holds if the maximum principle is satisfied.

Let us show that the splitting method of [7]-[9] preserves the maximum order of accuracy $O(h^2) + O(\tau^2)$ on arbitrary non-uniform nets Ω_0. Without attempting to generalise this method for the general equation (1.1_0) and arbitrary p let us give the argument for the case $c = 1$, $r_\alpha = 0$, $q = 0$, $f = 0$ and $p = 2$. Then we can write the scheme of [7] in the form

$$y_{t_1} = 0.5\,[\Lambda_1 (y^j + y^{j+1/2}) + \Lambda_2 y^j] + \frac{1}{4}\tau\Lambda_1\Lambda_2 y^j, \qquad y_{\bar{t}_2} = 0.5\Lambda_2 y^{j+1}, \tag{11}$$

where $\Lambda_\alpha y = (a_\alpha (x, t_{j+1/2})\, y_{\bar{x}_\alpha})_{\hat{x}_\alpha}$, a_α is given by the formulae of Section 1, Para. 3, and in this case k_α is continuous and $\tilde{k}_\alpha = k_\alpha$ (in [7] $a_\alpha = k_\alpha^{(-0.5_\alpha)}$ and the net ω_h is uniform).

The method is applicable to the parallelepiped $\bar{G} = \{0 \leqslant x_\alpha \leqslant l_\alpha\}$ and can be generalised (cf. [7]) to the case of regions formed from parallelepipeds. The second equation is also written for $x_1 = 0$, $x_1 = l_1$ and used to find the boundary values of $y^{j+1/2}$ for $x_1 = 0$, $x_1 = l_1$. Eliminating $y^{j+1/2}$ from (11) we obtain

$$y_{\bar{t}} = 0.5\,\Lambda\,(y^{j+1} + y^j) - \frac{\tau^2}{4}\Lambda_1\Lambda_2 y_{\bar{t}}. \tag{12}$$

Using this to find z we have

$$z_{\bar{t}} = 0.5\,\Lambda\,(z^{j+1} + z^j) - \frac{1}{4}\tau^2\Lambda_1\Lambda_2 z_{\bar{t}} + \Psi; \qquad z|_\gamma = 0, \quad z(x, 0) = 0, \tag{13}$$

where $\Psi = 0.5\Lambda(u^{j+1} + u^j) - \frac{\tau^2}{4}\Lambda_1\Lambda_2 u_{\bar{t}} - u_{\bar{t}}$. On a uniform net $\Psi = O(h^2) + O(\tau^2)$. On a non-uniform net Ω_0

$$\Psi = \sum_{\alpha=1}^{2} (\mu_\alpha)_{\hat{x}_\alpha} + \psi^*, \qquad \psi^* = \tilde{\psi} + O(\tau^2) + O(h^2), \quad \tilde{\psi} = -\frac{\tau^2}{4}\Lambda_1\Lambda_2 u_{\bar{t}}. \tag{14}$$

It is not difficult to see that $\tilde{\psi} = O(\tau^2)$ on any net $\bar{\Omega}_0$. The methods of [14] and [17] enable us to obtain the necessary *a priori* estimate for z without difficulty, with the condition that $|(a_\alpha)_{\bar{t}}| \leqslant c_1^*$, $c_1^* = \mathrm{const} > 0$. As usual, making a scalar multiplication of equation (12) by $2\tau z_{\bar{t}} H$, we obtain

$$2\tau \| z_{\bar t} \|^2 + I + 0.5\tau^3 (\Lambda_1\Lambda_2 z_{\bar t}, z_{\bar t}) = I^* + 2\tau (\Psi, z_{\bar t}) \tag{15}$$

$$I = I^{j+1} = \sum_{\alpha=1}^{2} (a_\alpha^{j+1/2}, (z_{\bar x_\alpha}^{j+1})^2]_\alpha, \quad I^* = \sum_{\alpha=1}^{2} (a_\alpha^{j+1/2}, (z_{\bar x_\alpha}^{j})^2]_\alpha, \quad \check I = I^j \text{ (see below)},$$

which differs from the usual identities considered by us previously in the presence of the term $0.5\tau^3(\Lambda_1\Lambda_2 z_{\bar t}, z_{\bar t})$. In [22], where an *a priori* estimate was obtained for the problem (13) on a uniform net this term is transformed by a double application of Green's formula in the directions x_1 and x_2. (The substitution becomes zero since $z_{\bar x_\alpha} = 0$ when $x_\beta = 0, x_\beta = l_\beta$ for $\beta \neq \alpha$.) The estimate of [22] is also valid in our case:

$$\tau^3 (\Lambda_1\Lambda_2 z_{\bar t}, z_{\bar t}) \geqslant 0.5 c_1^2 \tau^3 \| z_{\bar x_1 \bar x_2 \bar t} \|^2 - M\tau(I + \check I),$$

if $|(a_\alpha)_{\bar x_\beta}| \leqslant$ const. when $\alpha \neq \beta$.

Now let us transform $2\tau(\psi, z_{\bar t})$, using the fact that $\psi = \psi_1 + \psi_2$, $\psi_\alpha = (\mu_\alpha)_{\hat x_\alpha}$,

$$\tau((\mu_\alpha)_{\hat x_\alpha}, z_{\bar t}) = -\tau((\mu_\alpha, z_{\bar x_\alpha}]_\alpha)_{\bar t} + \tau((\mu_\alpha)_{\bar t}, \check z_{\bar x_\alpha}]_\alpha,$$

$$2\tau(\psi^*, z_{\bar t}) \leqslant \tau \| \psi^* \|^2 + \tau \| z_{\bar t} \|^2.$$

As a result we obtain an energy identity for $\|\tau\|_0 < \tau_0$:

$$(1 - M)\ I \leqslant (1 + M\tau)\check I - \sum_{\alpha=1}^{2} ((\mu_\alpha, z_{\bar x_\alpha})_\alpha)_{\bar t}\tau + M\tau(\|\mu_{\bar t}\|^2 + \|\psi^*\|^2 + \tau^4 \|\Lambda_1\Lambda_2 u_{\bar t}\| \tag{16}$$

where

$$\|\mu\| = \sum_{\alpha=1}^{2} \|\mu_\alpha\|_{2\alpha}.$$

By analogy with [12] and Section 2 we find from this

$$\| z(x, t) \| \leqslant M \{ \overline{\|\mu(x, t)\|} + \overline{\|\mu_{\bar t}(x, t)\|} + \overline{\|\psi^*(x, t)\|} + \overline{\tau^2 \|\Lambda_1\Lambda_2 u_{\bar t}\|} \} \tag{17}$$

for sufficiently small $\|\tau\|_0 < \tau_0$.

This proves that the scheme (11) retains second order accuracy on an arbitrary non-uniform net $\bar\Omega_0$:

$$\| y - u \| \leqslant M (\| \hbar^2 \| + \| \tau^2 \|_2) \text{ when } \|\tau\| < \tau_0. \tag{18}$$

We note that the smoothness requirements laid down for the solution $u = u(x, t)$ and the coefficients of the differential equation for which estimates (17) and (18) are valid augment as the number of dimensions increases (cf. [7]-[9], and compare with Para. 1, Section 3). The possibility of removing the restriction $|(a_\alpha)_{\bar{x}_\beta}| <$ const., whose presence makes it doubtful whether this method can be used in the case of discontinuous coefficients and quasi-linear equations when $k_\alpha = k_\alpha(x, t, u)$ is an interesting question.

Note. In [2] the scheme

$$y_{\bar{t}_1} = \Lambda_1 y^{j+1/2} + \Lambda_2 y^j, \qquad y_{\bar{t}_2} = \Lambda_2 y^{j+1} - \Lambda_2 y^j = \tau\Lambda_2 y_{\bar{t}}$$

of first order approximation is discussed. It is not difficult to show that that scheme

$$y_{\bar{t}_1} = 0.5\Lambda_1 y^{j+1/2} + y^j) + \Lambda_2 y^j, \qquad y_{\bar{t}_2} = 0.5\tau\Lambda_2 y_{\bar{t}}$$

leads to equation (12) and, therefore, has second order accuracy. The estimates (17) and (18) are valid for this scheme.

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