SCHEMES OF HIGH-ORDER ACCURACY FOR THE MULTI-DIMENSIONAL HEAT CONDUCTION EQUATION*

A. A. SAMARSKII

(Moscow)

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In [1] an economical scheme is put forward for the heat conduction equation

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} \frac{\partial^2 u}{\partial x_\alpha^2}, \qquad p \leqslant 4.$$

with accuracy $O(h^4 + \tau^2)$, where h is the step of the space net, and τ the time step. It is a three-layer scheme.

In this paper (Section 2-4) we examine two-layer schemes of $O(h^4 + \tau^2)$ (i.e. of order (4,2)) which are suitable for $p \leqslant 3$.** They are put into effect with the aid of a number of splitting algorithms or alternating direction algorithms which involve practically the same number of operations as the corresponding algorithms of $O(h^2 + \tau^2)$ (see [2]-[6]). It is shown that these schemes are absolutely stable and that for any values of $\gamma = \tau/h^2$ they converge in the mean at a rate $O(h^4 + \tau^2)$. In Section 5 we consider a three-layer scheme of order (4,2) which is more economical than the scheme of [1] and is suitable for $p \leqslant 4$. This scheme is absolutely stable and converges for any γ (the scheme of [1] converges for $\gamma \geqslant \gamma_0 =$ const. > 0).

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** Note added at proof stage. After sending this paper to the printer we succeeded in proving that our scheme is absolutely stable and converges in $L_2(\omega_h)$ at a rate $O(|h|)^4 + \tau^2)$. The corresponding proof will be published separately.

A scheme of order (4,2) for the equation

$$\frac{\partial u}{\partial t} = \sum_{\alpha,\beta=1}^{2} a_{\alpha\beta} \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta}, \qquad \sum_{\alpha,\beta=1}^{2} a_{\alpha\beta}\xi_\alpha\xi_\beta \geqslant c_1 \sum_{\alpha=1}^{2} \xi_\alpha^2, \qquad a_{\alpha\beta} = \text{const.}$$

is given in Section 6.

In Section 7 we examine two-layer schemes of the same order of accuracy for the equation with variable coefficients,

$$c(x, t)\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x, t)\frac{\partial u}{\partial x_\alpha}\right) + f(x, t), \qquad x = (x_1, \ldots, x_p), \quad p \leqslant 2.$$

1. A scheme of high-order accuracy for the one-dimensional heat conduction equation

1. Let the problem

$$\frac{\partial u}{\partial t} = Lu + f, \qquad Lu = \frac{\partial^2 u}{\partial x^2}, \qquad f = f(x, t), \tag{1}$$

$$u(0, t) = u_1(t), \qquad u(l, t) = u_2(t); \qquad u(x, 0) = u_0(x). \tag{2}$$

be given in the rectangle $\bar{D} = (0 \leqslant x \leqslant l, 0 \leqslant t \leqslant T)$.

Let us introduce the difference net $\bar{\Omega} = \bar{\omega}_h \times \bar{\omega}_\tau = \{(x_i, \ t_j) \in \bar{D}\}$, where $\bar{\omega}_h = \{x_i = ih, 0 \leqslant i \leqslant N, \ \ h = l/N\}$, $\bar{\omega}_\tau = \{t_j, \ \ 0 \leqslant j \leqslant K, t_0 = 0, t_K = T\}$. The net $\bar{\omega}_h$ is uniform, and $\bar{\omega}_\tau$ is an arbitrary non-uniform time net with step $\tau_{j+1} = t_{j+1} - t_j$. We shall use the notation of [7] putting

$$y = y_i^{j+1} = y(x_i, t_{j+1}), \qquad \check{y} = y^j, \quad y^{(\pm 1)} = y(x_i \pm h, t_{j+1}), \qquad \tau = \tau_{j+1},$$
$$y_{\bar{x}} = (y - y^{(-1)})/h, \qquad y_x = (y^{(+1)} - y)/h, \qquad y_{\bar{t}} = (y - \check{y})/\tau, \qquad \Lambda y = y_{\bar{x}x}.$$

To write down the high accuracy scheme we use the asymptotic expansion

$$\Lambda u = u_{\bar{x}x} = Lu + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4} + O(h^4) = Lu + \frac{h^2}{12} L\left(\frac{\partial u}{\partial t} - f\right) + O(h^4),$$

where u is a solution of equation (1). It follows that

$$0.5\Lambda(u + \check{u}) = (Lu)^{j+1/2} + \frac{h^2}{12}\Lambda(u_{\bar{t}} - f) + O(h^4) + O(\tau^2),$$

where $u^{j+1/2} = u(x, t_{j+1/2})$, $t_{j+1/2} = 0.5(t_j + t_{j+1})$, i.e. the scheme

$$y_{\bar{t}} = 0.5\Lambda(y + \check{y}) - \frac{h^2}{12}\Lambda y_{\bar{t}} + \varphi, \qquad (3)$$
$$\varphi = \left(f + \frac{h^2}{12}\Lambda f\right)^{j+1/2} = \left\{\frac{1}{12}(f^{(+1)} + f^{(-1)}) + \frac{5}{6}f\right\}^{j+1/2}$$

has fourth order approximation w.r.t. h and second order approximation w.r.t. τ.

The scheme (3) is usually written in the form

$$y_{\bar{t}} = \sigma\Lambda y + (1 - \sigma)\Lambda\check{y} + \varphi; \quad y_0 = u_1, \quad y_N = u_2; \quad y(x, 0) = u_0(x), \qquad (4)$$

$$\sigma = 0.5\left(1 - \frac{h^2}{6\tau}\right) = 0.5\left(1 - \frac{1}{6\gamma}\right), \qquad \gamma = \tau/h^2. \qquad (5)$$

To find $y = y^{j+1}$ we have the problem:

$$\tau\sigma\Lambda y - y = -F, \quad y_0 = u_1, \quad y_N = u_2; \quad F = \check{y} + \tau(1 - \sigma)\Lambda\check{y} + \tau\varphi,$$

or

$$\sigma\gamma y_{i-1} - (1 - 2\sigma\gamma)\, y_i + \sigma\gamma y_{i+1} = -F_i, \quad y_0 = u_1, \quad y_N = u_2. \qquad (6)$$

The solution of this problem for any values of γ, i.e. for $-\infty < \sigma < 0.5$, can be found with the known formulae of successive substitution [8], [9]:

$$\alpha_{i+1} = \frac{\sigma\gamma}{1 + \sigma\gamma + \sigma\gamma(1 - \alpha_i)}, \quad \alpha_1 = 0, \quad i = 1, 2, \ldots, N-1,$$
$$\beta_{i+1} = \frac{\sigma\gamma\beta_i + F_i}{1 + \sigma\gamma + \sigma\gamma(1 - \alpha_i)}, \quad \beta_1 = u_1, \quad i = 1, 2, \ldots, N-1,$$
$$y_i = \alpha_{i+1}y_{i+1} + \beta_{i+1}, \quad y_N = u_2, \quad i = 1, 2, \ldots, N = 1.$$

Computation with these formulae is always stable, since

$$\sigma\gamma = 0.5\gamma - \frac{1}{12} > -\frac{1}{12}.$$

Note. In the book [9] the schemes 6 and 12 are considered separately on pp. 108 and 109. It is not difficult to see that both these schemes are algebraically identical to scheme (4).

2. In order to discover the order of accuracy of the scheme (4) we have to find estimates for the solution of the problem

$$z_{\bar{t}} = \sigma\Lambda z + (1 - \sigma)\Lambda\check{z} + \psi, \quad z_0 = z_N = 0, \quad z(x, 0) = 0, \qquad (7)$$

where $z = y - u$, u is a solution of the problem (1)-(2), y is a solution of the problem (4), $\psi = \sigma\Lambda u + (1-\sigma)\Lambda\check{u} + \varphi - u_{\bar{t}}$ is the approximation error of the scheme (4).

We have not succeeded in discovering any published proof of the uniform convergence at a rate $O(h^4 + \tau^2)$ of scheme (4). We therefore thought it desirable to give this proof here, despite its elementary nature. Moreover we use a similar method in Section 4 when studying the convergence of multidimensional schemes of the same order of accuracy. In [10], where an attempt was made to obtain the corresponding estimate, there is an error which was pointed out to us by V.B. Andreyev (the condition $2\eta - 1 > 0$ in Theorem 5 should be $2\eta - 1 < 0$ or $\sigma > 0.5$ since $\eta = 1 - \sigma$).

3. Let us consider the more general problem

$$z_{\bar{t}} = \Lambda z - \tau\eta\Lambda z_{\bar{t}} + \psi, \qquad z_0 = z_N = 0, \qquad z(x, 0) = z^0(x), \tag{8}$$

where η is an arbitrary parameter, and let us find the conditions for which the scheme (8) is absolutely stable w.r.t. to the initial data and the right-hand side. Inserting

$$\eta = 1 - \sigma = 0.5\left(1 + \frac{1}{6\gamma}\right), \tag{9}$$

in (8) we obtain the scheme (4).

We need the notation of [7]:

$$(y, v) = \sum_{i=1}^{N-1} y_i v_i h, \qquad (y, v] = \sum_{i=1}^{N} y_i v_i h, \qquad \|y\|_0 = \max_{\bar{\omega}_h} |y_i|,$$
$$\|v\| = \sqrt{(v, v)}, \qquad \|v_{\bar{x}}\| = \sqrt{(1, v_{\bar{x}}^2]}$$

(y, v are arbitrary functions given on $\bar{\omega}_h$) together with Green's difference formula

$$(y, v_x) = -(v, y_{\bar{x}}], \qquad \text{if} \quad y_0 = y_N = 0.$$

Lemma 1.

$$\|v_{\bar{x}}\|^2 \leqslant \frac{4}{h^2}\|v\|^2, \tag{10}$$

where v is an arbitrary net function, given on $\bar{\omega}_h$, $v_0 = v_N = 0$.

For $h^2\|v_{\bar{x}}\|^2 = (v - v^{(-1)},\ v - v^{(-1)}] \leqslant 2(v^2 + (v^{(-1)})^2,\ 1]$.

Let us write down the energy identity for (8). Multiplying equation

(8) scalarly by $\tau z_{\bar{t}}$ and using the relation

$$vv_{\bar{t}} = 0.5\,(v^2)_{\bar{t}} + 0.5\tau v_{\bar{t}}^2 \qquad (v_{\bar{t}}^2 = (v_{\bar{t}})^2), \tag{11}$$

we obtain

$$\tau \| z_{\bar{t}} \|^2 + 0.5\tau\,(\| z_{\bar{x}} \|^2)_{\bar{t}} = (\eta - 0.5)\,\tau \| z_{\bar{x}\bar{t}} \|^2 + \tau\,(\psi,\, z_{\bar{t}}).$$

If $\eta \leqslant 0.5$ $(\sigma \geqslant 0.5)$, then $\eta - 0.5 \leqslant 0$ and the corresponding term can be omitted if we replace the sign = by $\leqslant$. This case has been extensively studied. Let us therefore consider the case $\eta \geqslant 0.5$ or $\sigma \leqslant 0.5$. Inserting $v = z_{\bar{t}}$ in Lemma 1 we find

$$\tau\,[1 - 4\,(\eta - 0.5)\,\gamma]\,\| z_{\bar{t}} \|^2 + 0.5\tau\,(\| z_{\bar{x}} \|^2)_{\bar{t}} \leqslant \tau\,(\psi,\, z_{\bar{t}}). \tag{12}$$

Theorem 1. If the condition

$$\eta \leqslant 0.5 + \frac{1}{4\gamma}\,(1 - \varepsilon), \qquad \varepsilon = \text{const.} > 0, \tag{13}$$

is satisfied, then scheme (8) is absolutely stable for any h and τ:

$$\frac{2}{\sqrt{l}}\,\| z^{j+1} \|_0 \leqslant \| z_{\bar{x}}^{j+1} \| \leqslant \| z_{\bar{x}}\,(x,\, 0) \| + \frac{1}{\sqrt{2\varepsilon}}\,\overline{\| \psi^{j+1} \|}, \tag{14}$$

where

$$\overline{\| \psi^{j+1} \|} = \left(\sum_{j'=1}^{j+1} \tau_{j'}\,\| \psi^{j'} \|^2 \right)^{1/2}. \tag{15}$$

If

$$\eta \leqslant 0.5 + \frac{1}{4\gamma}, \tag{16}$$

then

$$\| z_{\bar{x}}^{j+1} \| \leqslant \| z_{\bar{x}}\,(x,\, 0) \| + M\,(\| \psi^{j+1} \| + \overline{\| \psi_{\bar{t}}^{j+1} \|}), \tag{17}$$

where M is a constant which depends only on T and l.

The estimates (14) and (17) follow from inequality (12). To prove (14) we must use the estimate

$$\tau\,(\psi,\, z_{\bar{t}}) \leqslant \varepsilon\tau\,\| z_{\bar{t}} \|^2 + \frac{\tau}{4\varepsilon}\,\| \psi \|^2$$

and (13). In the case of (16) we have

$$\tau\,(\psi,\, z_{\bar{t}}) = \tau\,(\psi,\, z)_{\bar{t}} - \tau\,(\psi_{\bar{t}},\, \check{z}) \leqslant \tau\,(\psi,\, z)_{\bar{t}} + c_0\tau\,\| \check{z}_{\bar{x}} \|^2 + \frac{\tau l^2}{4c_0}\,\| \psi_{\bar{t}} \|^2,$$

since

$$\| z \|_0 \leqslant \frac{\sqrt{l}}{2} \| z_{\bar{x}} \|. \tag{18}$$

The subsequent argument is similar to that of [7].

4. Let us turn to problem (6). Comparing (6) and (8) we can see that $\eta = 0.5 + 1/12\,\gamma$ for the scheme (6), i.e. condition (13) is satisfied. Theorem 1 (for $\varepsilon = 2/3$) gives

$$\| z^{j+1} \|_0 \leqslant \frac{\sqrt{3l}}{4} \overline{\| \psi^{j+1} \|}. \tag{19}$$

We have proved the following theorem.

Theorem 2. If the solution $u = u(x, t)$ of the problem (1)-(2) satisfies the conditions for which the scheme (4) has maximum order of approximation

$$\| \psi \| \leqslant M\ (h^4 + \tau^2),$$

then scheme (3) converges uniformly at a rate $O(h^4 + \tau^2)$, so that on an arbitrary sequence of nets $\bar{\Omega}$ we have the estimate

$$\| y\ (x,\ t_{j+1}) - u\ (x,\ t_{j+1}) \|_0 \leqslant M'\ (h^4 + \| \tau^2 \|_{j+1}), \qquad M' = \frac{\sqrt{3l}}{4} M,$$

where M is a positive constant not depending on the net, and

$$\| \tau^2 \|_{j+1} = \left(\sum_{j'=1}^{j+1} (\tau_{j'}^2)^2\ \tau_{j'} \right)^{1/2}.$$

2. Schemes of high-order accuracy for the heat conduction equation with several space variables

1. Let us consider the heat conduction equation with constant coefficients:

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} \varkappa_\alpha \frac{\partial^2 u}{\partial x_\alpha^2} + f\ (x,\ t),$$

where $x = (x_1, \ldots, x_p)$ is a point of p-dimensional Euclidean space. Without loss of generality we can take $\varkappa_\alpha = 1$, since this can always be arranged by introducing the new variables $x'_\alpha = x_\alpha / \sqrt{\varkappa_\alpha}$. Let

$\overline{G} = \{0 \leqslant x_\alpha \leqslant l_\alpha, \quad \alpha = 1, 2, \ldots, p\}$ be a p-dimensional parallelepiped and Γ its boundary, so that $\overline{G} = G + \Gamma$. We put $Q_T = G \times (0 < t \leqslant T]$, $\overline{Q}_T = \overline{G} \times [0 \leqslant t \leqslant T]$. In the cylinder $\overline{Q_T}$ we look for a solution of the first boundary-value problem:

$$\frac{\partial u}{\partial t} = Lu + f(x, t), \qquad Lu = \sum_{\alpha=1}^{p} L_\alpha u, \qquad L_\alpha u = \frac{\partial^2 u}{\partial x_\alpha^2}, \qquad (x, t) \in Q_T, \tag{1}$$

$$u|_\Gamma = \mu(x, t), \quad 0 \leqslant t \leqslant T; \qquad u(x, 0) = u_0(x), \quad x \in \overline{G}. \tag{2}$$

Let us introduce difference nets. Let $\overline{\omega}_h = \{x_i \in \overline{G}\}$ be a space net; here $x_i = (i_1 h_1, \ldots, i_p h_p)$, $i_\alpha = 0, 1, \ldots, N_\alpha$, $\alpha = 1, \ldots, p$, $h_\alpha = l_\alpha / N_\alpha$. The net $\overline{\omega}_h$ is uniform only w.r.t. each of the space variables, the steps h_α and h_β in general being different when $\alpha \neq \beta$. Let $\omega_h = \{x_i \in G\}$ be the set of internal nodes, $\gamma = \{x_i \in \Gamma\}$ the set of boundary nodes, γ_α^- the set of nodes of the boundary γ for which $x_\alpha = 0$ and γ_α^+ the set of nodes of γ for which $x_\alpha = l_\alpha$.

The net $\overline{\omega}_\tau = \{t_j \in [0 \leqslant t \leqslant T], \ t_0 = 0, \ t_K = T, \ 0 \leqslant j \leqslant K\}$ is non-uniform, its step $\tau_{j+1} = t_{j+1} - t_j > 0$ only satisfying the normalisation condition

$$\sum_{j'=1}^{K} \tau_{j'} = T.$$

We put $\omega_\tau = \{t_j \in (0 < t \leqslant T]\}$, $\overline{\Omega} = \overline{\omega}_h \times \overline{\omega}_\tau$, $\Omega = \omega_h \times \omega_\tau$. Following [11], [12] we use the notation $v = v(x_i, t_{j+1}) = v(x, t) = v^{j+1}$, $\check{v} = v^j$, $\tau = \tau_{j+1}$, $\check{\tau} = \tau_j$, $v^{(\pm 1_\alpha)} = v(x^{(\pm 1_\alpha)}, t)$, $x^{(\pm 1_\alpha)} = (i_1 h_1, \ldots, i_{\alpha-1} h_{\alpha-1}, (i_\alpha \pm 1) h_\alpha, i_{\alpha+1} h_{\alpha+1}, \ldots, i_p h_p)$, $v_{\bar{x}_\alpha} = (v - v^{(-1_\alpha)})/h_\alpha$, $v_{x_\alpha} = (v^{(+1_\alpha)} - v)/h_\alpha$, $v_{\bar{t}} = (v - \check{v})/\tau$; $\quad |h| = \sqrt{h_1^2 + \cdots + h_p^2}$.

2. Let us go on to derive schemes which have fourth order approximation w.r.t. $|h|$ and second order approximation w.r.t. τ (schemes of order (4,2)). By analogy with Section 1, Para. 1 we have

$$\Lambda_\alpha u = u_{\bar{x}_\alpha x_\alpha} = L_\alpha u + \frac{h_\alpha^2}{12} L_\alpha L_\alpha u + O(h_\alpha^4).$$

Inserting

$$L_\alpha u = \frac{\partial u}{\partial t} - \sum_{\beta \neq \alpha} L_\beta u - f,$$

from (1) we find

$$\Lambda_\alpha u = L_\alpha u + \frac{h_\alpha^2}{12} L_\alpha \frac{\partial u}{\partial t} - \frac{h_\alpha^2}{12} \sum_{\beta \neq \alpha} L_\alpha L_\beta u - \frac{h_\alpha^2}{12} L_\alpha f + O(h_\alpha^4).$$

It follows that the operator

$$0.5\,\Lambda\,(u + \check{u}) - \frac{1}{12} \sum_{\alpha=1}^{p} h_\alpha^2 \Lambda_\alpha u_{\bar{t}} + \frac{1}{12} \sum_{\alpha=1}^{p} h_\alpha^2 \left\{ \sum_{\beta \neq \alpha} \Lambda_\alpha \Lambda_\beta \check{u} - \Lambda_\alpha f \right\}$$

has approximation error $O(|h|^4 + \tau^2)$ in the class of sufficiently smooth solutions $u = u(x, t)$ of equation (1) w.r.t. the operator $(Lu)^{j+1/2}$. Therefore the difference scheme

$$y_{\bar{t}} = 0.5\Lambda\,(y + \check{y}) - \frac{1}{12} \sum_{\alpha=1}^{p} h_\alpha^2 \Lambda_\alpha y_{\bar{t}} + \frac{1}{12} \sum_{\alpha=1}^{p} h_\alpha^2 \sum_{\beta \neq \alpha}^{1-p} \Lambda_\alpha \Lambda_\beta \check{y} + \varphi, \qquad (3)$$

$$\varphi = \left(f + \frac{1}{12} \sum_{\alpha=1}^{p} h_\alpha^2 \Lambda_\alpha f\right)^{j+1/2}, \quad \Lambda = \sum_{\alpha=1}^{p} \Lambda_\alpha, \qquad (4)$$

for the solution $u = u(x, t)$ has approximation error

$$\psi = O(|h|^4 + \tau^2).$$

We introduce the notation

$$\sigma_\alpha = \tfrac{1}{2}(1 - h_\alpha^2/6\tau) \qquad (5)$$

and use the relation $h_\alpha^2/12\tau = 0.5 - \sigma_\alpha$. Then we can write the scheme (3) in the form

$$y_{\bar{t}} = \sum_{\alpha=1}^{p} [\sigma_\alpha \Lambda_\alpha y + (1 - \sigma_\alpha) \Lambda_\alpha \check{y}] + \tau R_p \check{y} + \varphi, \qquad (6)$$

$$y = \mu \quad \text{for } x \in \gamma, \quad t \in \bar{\omega}_\tau; \qquad y(x, 0) = u_0(x) \quad \text{for } x \in \bar{\omega}_h, \qquad (7)$$

where

$$R_p y = \sum_{\alpha=1}^{p} \sum_{\beta > \alpha} \frac{h_\alpha^2 + h_\beta^2}{12\tau} \Lambda_\alpha \Lambda_\beta\, y = \sum_{\alpha=1}^{p} \sum_{\beta=\alpha+1}^{p} (1 - \sigma_\alpha - \sigma_\beta) \Lambda_\alpha \Lambda_\beta y, \qquad (8)$$

$$R_2 y = (1 - \sigma_1 - \sigma_2) \Lambda_1 \Lambda_2 y, \qquad R_3\, y = (1 - \sigma_1 - \sigma_2) \Lambda_1 \Lambda_2 y + \qquad (9)$$
$$+ (1 - \sigma_1 - \sigma_3) \Lambda_1 \Lambda_3 y + (1 - \sigma_2 - \sigma_3) \Lambda_2 \Lambda_3 y.$$

Inserting $\sigma_\alpha = 0.5$ formally, from (6) we obtain a symmetric second order scheme, and for $\sigma_\alpha = 1$ a purely implicit scheme.

Below we call (6) the *initial* scheme.

3. Following [2]-[6] we form a scheme with a split operator on the top row which has the same order of approximation as the scheme (6). We call this scheme the generating scheme. When $p = 2$ the generating scheme obviously has the form

$$(E - \tau\sigma_1\Lambda_1)(E - \tau\sigma_2\Lambda_2)\,y = (E + (1 - \sigma_1)\,\tau\Lambda_1)(E + \tau(1 - \sigma_2)\,\Lambda_2)\check{y} + \tau\varphi,$$

where E is the unit operator, or

$$y_{\bar{t}} = \sum_{\alpha=1}^{2} [\sigma_\alpha\Lambda_\alpha\, y + (1 - \sigma_\alpha)\,\Lambda_\alpha\check{y}] + \tau R_2\check{y} - \tau^2\sigma_1\sigma_2\Lambda_1\Lambda_2 y_{\bar{t}} + \varphi. \qquad (10)$$

Comparing (6) and (10) we see that the generating and the initial schemes have the same order of approximation in the class of sufficiently smooth solutions of equation (1).

When $p = 3$ we choose the generating scheme

$$\prod_{\alpha=1}^{3} (E - \tau\sigma_\alpha\Lambda_\alpha)y = [E + \tau\sum_{\alpha=1}^{3} (1 - \sigma_\alpha)\,\Lambda_\alpha + \tau^2 R_3 + \tau^2 Q_3 - \tau^3\sigma_1\sigma_2\sigma_3\Lambda_1\Lambda_2\Lambda_3]\,\check{y} + \tau\varphi, \qquad (11)$$

where R_3 is given by formula (9) and

$$Q_3 = \sigma_1\sigma_2\Lambda_1\Lambda_2 + \sigma_1\sigma_3\Lambda_1\Lambda_3 + \sigma_2\sigma_3\Lambda_2\Lambda_3 = \sum_{\alpha=1}^{3}\sum_{\beta>\alpha} \sigma_\alpha\sigma_\beta\Lambda_\alpha\Lambda_\beta. \qquad (12)$$

Let us rewrite (11) in the form

$$y_{\bar{t}} = \sum_{\alpha=1}^{3} [\sigma_\alpha\Lambda_\alpha y + (1 - \sigma_\alpha)\,\Lambda_\alpha\check{y}] + \tau R_3\check{y} - \tau^2 Q_3 y_{\bar{t}} + \tau^3\sigma_1\sigma_2\sigma_3\Lambda_1\Lambda_2\Lambda_3 y_{\bar{t}} + \varphi. \qquad (13)$$

It is not difficult to see that its maximum order of approximation in the class of solutions of equation (1) is $O(|h|^4 + \tau^2)$. Together with (13) we shall consider the simpler generating scheme

$$y_{\bar{t}} = \sum_{\alpha=1}^{3} [\sigma_\alpha\Lambda_\alpha y + (1 - \sigma_\alpha)\,\Lambda_\alpha\check{y}] + \tau R_3\check{y} - \tau^2 Q_3 y_{\bar{t}} + \tau^2\sigma_1\sigma_2\sigma_3\Lambda_1\Lambda_2\Lambda_3 y + \varphi. \qquad (14)$$

Thus we have put the initial problem (1)-(2) in correspondence with the difference problem

$$\mathscr{L}_p\,[y] = 0 \quad \text{for} \quad (x, t) \in \Omega, \qquad (15)$$

$$y = \mu\,(x, t) \ \text{for} \ x \in \gamma, \quad t \in \bar{\omega}_\tau; \quad y\,(x, 0) = u_0\,(x) \ \text{for}\ x \in \bar{\omega}_h, \qquad (16)$$

where $\mathscr{L}_2\,[y] = 0$ is equation (10), $\mathscr{L}_3\,[y] = 0$ is equation (13) or (14).

3. Computational alternating direction algorithms

1. The solution of the multidimensional problem (2.15)-(2.16) can be reduced to the successive solution of one-dimensional algebraic problems w.r.t. the directions $x_1, \ldots, x_p$ (see [2]-[6], [15]). This reduction is usually called the splitting (factorisation) of a multidimensional operator into one-dimensional operators, or simply splitting. Several computational alternating direction algorithms correspond to one generating scheme. Thus, for instance, it is not difficult to see that the algorithms of [3] and [5] correspond to the one generating scheme which is a special case of scheme (2.13) for $\sigma_\alpha = 0.5$ and, therefore, $R_p = 0$. Let us find another algorithm for this scheme. Putting $y = \check{y} + \tau y_{\bar{t}}$, we write this generating scheme in the form

$$\prod_{\alpha=1}^{p} A_\alpha y_{\bar{t}} = \Lambda \check{y} + \varphi, \qquad A_\alpha = E - 0.5\tau\Lambda_\alpha. \tag{1}$$

From this we have the algorithm

$$\text{Б}'. \qquad A_1 v_{(1)} = \Lambda\check{y} + \varphi, \quad A_\alpha v_{(\alpha)} = v_{(\alpha-1)}, \qquad \alpha > 1, \quad v_{(p)} = y_{\bar{t}}, \tag{2}$$
$$y = \check{y} + \tau v_{(p)}.$$

In accordance with [5] to find $v_{(\alpha)}$ for $x \in \gamma_\alpha$, $1 \leqslant \alpha < p$, we must use the formulae

$$v_{(\alpha)} = A_{\alpha+1} \ldots A_p y_{\bar{t}}, \qquad y_{\bar{t}} = (y^{j+1} - y^{j})/\tau = (\mu_\alpha)_{\bar{t}}^{j+1},$$
$$\mu_\alpha = \mu \quad \text{for} \quad x \in \gamma_\alpha. \tag{3}$$

This algorithm is more economical that those of [2], [15]. In particular, it is suitable for the solution of the stationary equation

$$\Lambda y + \varphi = 0,$$

by the iterative method, since in this case $y_{\bar{t}} = 0$ for $x \in \gamma$ and we always obtain the homogeneous boundary condition $v_{(\alpha)} = 0$ for $v_{(\alpha)}$ for $x \in \gamma_\alpha$, $\alpha = 1, \ldots, p$.

2. For the generating scheme (2.15) we use the algorithms given in [2]-[6].

Let us first consider the two-dimensional problem ($p = 2$). We introduce the fractional step $t_{j+1/2}$ and put $y_{(1)} = y^{j+1/2}$ the corresponding value of the unknown function y. Let us construct three one-dimensional algorithms (see [2], [3], [5]).

$$\text{A.}\qquad y_{\bar{t}_1}=\frac{y_{(1)}-\check{y}}{\tau}=\sigma_1\Lambda_1 y_{(1)}+F[\check{y}],\qquad y_{\bar{t}_2}=\frac{y_{(2)}-y_{(1)}}{\tau}=\sigma_2\Lambda_2 y_{(2)},\tag{4}$$

$$F[\check{y}]=\sum_{\alpha=1}^{2}(1-\sigma_\alpha)\Lambda_\alpha\check{y}+(1-\sigma_1)(1-\sigma_2)\tau\Lambda_1\Lambda_2\check{y}.\tag{5}$$

The boundary conditions (2.16) are given only for $\check{y}$ and y. According to [5] we can find the boundary values for $y_{(1)}$ from the formula

$$y_{(1)}=A_2 y \text{ for } x_1=0,\qquad x_1=l_1.\tag{6}$$

The order of the computation is as follows: we find $y_{(1)}$ for $x\in\gamma_1$ from (6), calculate $F[\check{y}]$ from (5) and then, using the one-dimensional successive substitution formulae (see Section 1), solve successively equations (4).

B.

$$y_{\bar{t}_1}=\sigma_1\Lambda_1 y_{(1)}+(1-\sigma_1)\Lambda_1\check{y}+\frac{1-\sigma_2}{\sigma_1}\Lambda_2\check{y}+\varphi,\tag{7}$$

$$y_{\bar{t}_2}=\sigma_2\Lambda_2 y_{(2)}-\frac{(1-\sigma_1)(1-\sigma_2)}{\sigma_1}\Lambda_2\check{y},\qquad y_{(2)}=y=y^{j+1},\tag{8}$$

$$y_{(1)}=A_2 y+\frac{(1-\sigma_1)(1-\sigma_2)}{\sigma_1}\tau\Lambda_2\check{y}\text{ for } x_1=0,\quad x_1=l_1\quad (x\in\gamma_1).\tag{9}$$

The order of the computation is the same as for A. We can see from (7)-(9) that the scheme has no meaning for $\sigma_1=0$. Unlike in A, when finding y two layers ($\check{y}$ and $y_{(1)}$) must be used. Inserting $\sigma_1=\sigma_2=0.5$ formally in (7)-(9) we obtain a scheme of $O(|h|^2+\tau^2)$, and for $\sigma_1=\sigma_2=1$ a scheme $O(|h|^2+\tau)$ (see [2], [3]).

Let us suppose that $u\equiv 0$ for $x_1=0$, $x_1=l_1$. This can always be arranged by subtracting a function which is linear in x_1 and equal to $u(x,t)$ for $x_1=0$, $x_1=l_1$ from $u(x,t)$. In this case when $p=2$ the problem (15)-(16) is equivalent to the problem

$$\text{C.}\qquad y_{\bar{t}_1}=\sigma_1\Lambda_1 y_{(1)}+(1-\sigma_1)\Lambda_1\check{y},\qquad y_{(1)}=0\text{ for } x_1=0,\quad x_1=l_1,\tag{10}$$

$$y_{\bar{t}_2}=\sigma_2\Lambda_2 y_{(2)}+(1-\sigma_2)\Lambda_2 y_{(1)}+\Phi,\qquad y_{(2)}=\mu^{j+1}\quad\text{for}\quad x_2=0,\quad x_2=l_2,\tag{11}$$

$$A_1\Phi=\Phi-\sigma_1\tau\Lambda_1\Phi=\varphi,\qquad \Phi=0\text{ for } x_1=0,\quad x_1=l_1.\tag{12}$$

The equation (11) can be solved in the region $(0<x_1<l_1,\ 0\leqslant x\leqslant l_2)$, the value of $y_{(1)}$ found for $x_2=0$, $x_2=l_2$ being used to

solve problem (26). The order of the computation is as follows: problem (10) on the whole net $\bar{\omega}_h \to$ problem (12) on the net $\omega_h \to$ problem (11) on ωh.

Eliminating $y_{(1)}$, it is not difficult to see that all the algorithms A, B and C are equivalent to the generating scheme (2.15)-(2.16) for $p = 2$. Putting $\sigma_1 = \sigma_2 = 0.5$ we obtain known algorithms for schemes of accuracy $O(|h|^2 + \tau^2)$. In our case $\sigma_\alpha = 0.5\,(1 - h_\alpha^2/6\tau)$. The amount of computation for schemes of order $O(|h|^2 + \tau^2)$ and $O(|h|^4 + \tau^2)$ is practically the same. The increased order of accuracy is achieved purely by a corresponding choice of the parameters σ_1 and σ_2.

3. One-dimensional alternating direction algorithms for a three-dimensional generating scheme ($p = 3$) are constructed similarly. Let us write down algorithms A and B only. We introduce two fractional steps $t_{j+1/3}$ and $t_{j+2/3}$ and the corresponding values of $y_{(1)}$ and $y_{(2)}$, putting $y_{(3)} = y = y^{j+1}$, $\check{y} = y^j$.

A. $$y_{\bar{t}_1} = \sigma_1\Lambda_1 y_{(1)} + F[\check{y}], \qquad y_{\bar{t}\alpha} = \frac{y_{(\alpha)} - y_{(\alpha-1)}}{\tau} = \sigma_\alpha\Lambda_\alpha y_{(\alpha)}, \qquad \alpha > 1, \quad (13)$$

$$F[\check{y}] = F_1[\check{y}] = \sum_{\alpha=1}^{3}(1-\sigma_\alpha)\Lambda_\alpha\check{y} + \tau(R_3 + Q_3)\check{y} + \varphi \quad \text{for the scheme } (2.14), \quad (14)$$

$$F[\check{y}] = F_2[\check{y}] = F_1[\check{y}] - \tau^2\sigma_1\sigma_2\sigma_3\Lambda_1\Lambda_2\Lambda_3\check{y} \text{ for the scheme } (2.13), \quad (15)$$

$$\begin{aligned} y_{(1)} &= A_2A_3y \quad \text{for} \quad x_1 = 0, \quad x_1 = l_1, \\ y_{(2)} &= A_3y \quad \text{for} \quad x_2 = 0, \quad x_2 = l_2. \end{aligned} \quad (16)$$

B′.
$$A_1 v_{(1)} = \Lambda\check{y} + \tau R_3\check{y} + \varphi, \quad (17)$$

$$A_\alpha v_\alpha = v_{(\alpha-1)} \text{ or } v_{\bar{t}_\alpha} = \sigma_\alpha\Lambda_\alpha v_{(\alpha)}, \quad \alpha = 2,3; \quad v_{(3)} = y_{\bar{t}}, \quad (18)$$

$$y = \check{y} + \tau v_{(3)}. \quad (19)$$

The boundary conditions (16) must be added to these formulae. Eliminating $v_{(1)}$ and $v_{(2)}$ from (17)-(18) we obtain the scheme

$$A_1A_2A_3y_{\bar{t}} = \Lambda\check{y} + \tau R_3\check{y} + \varphi, \qquad A_\alpha = E - \tau\sigma_\alpha\Lambda_\alpha,$$

for $v_{(3)} = y_{\bar{t}}$. This is the same as the generating scheme (2.13), as we can see by making obvious transformations.

Algorithm B′ is a three-layer algorithm. When finding $v_{(\alpha)}$ we have to use not only $v_{(\alpha-1)}$, but also the values of $\check{y}$. However algorithm (17)-(19) is more economical in its number of operations than the three-layer scheme obtained in [1] for $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$. For $\sigma_\alpha = 0.5$, $\alpha = 1, 2, 3$ ($R_3 = 0$) we obtain from (17)-(19) an algorithm which gives an accuracy $O(|h|^2 + \tau^2)$ (see [2], [15], [12]).

4. Stability and convergence

1. Let us show that all the algorithms given in Section 3 are absolutely stable on an arbitrary net $\overline{\Omega}$ and in fact have accuracy $O(|h|^4 + \tau^2)$. To do this we must turn to the generating scheme (2.15)-(2.16). We note that stability and convergence were considered in [1] for the initial scheme. This is not sufficient to justify the method. Let y be a solution of the problem (2.15)-(2.16), and let $u = u(x, t)$ be a solution of the problem (2.1)-(2.2). Putting $y = z + u$ in (2.15)-(2.16) we obtain conditions for the net function $z = y - u$:

$$z_{\bar{t}} = \sum_{\alpha=1}^{p} [\sigma_\alpha \Lambda_\alpha z + (1 - \sigma_\alpha) \Lambda_\alpha \check{z}] + \tau R_p \check{z} - \tau^2 Q_p z_{\bar{t}} + \delta_{p,3} \tau^3 \sigma_1 \sigma_2 \sigma_3 \Lambda_1 \Lambda_2 \Lambda_3 z_{\bar{t}} + \Psi, \tag{1}$$

$$z = 0 \text{ for } x \in \gamma, \quad t \in \overline{\omega}_\tau; \qquad z(x, 0) = 0 \text{ for } x \in \overline{\omega}_h, \tag{2}$$

$$R_p = \sum_{\alpha=1}^{p} \sum_{\beta>\alpha} (1 - \sigma_\alpha - \sigma_\beta) \Lambda_\alpha \Lambda_\beta, \tag{3}$$

$$Q_p = \sum_{\alpha=1}^{p} \sum_{\beta>\alpha} \sigma_\alpha \sigma_\beta \Lambda_\alpha \Lambda_\beta \tag{4}$$

where $\delta_{p,3}$ is the Kroneker symbol, and Ψ the approximation error of the generating scheme,

$$\Psi = \psi - \tau^2 Q_p u_{\bar{t}} + \delta_{p,3} \tau^3 \sigma_1 \sigma_2 \sigma_3 \Lambda_1 \Lambda_2 \Lambda_3 u_{\bar{t}}, \tag{5}$$

where ψ is the approximation error of the initial scheme.

Let $C_{(n)}^{(m)}$ be the class of functions having derivatives w.r.t. x_α, $\alpha = 1, \ldots, p$ up to and including the m-th order and w.r.t. t up to and including the n-th order, bounded in $\overline{Q}_t$.

If $u = u(x, t) \in C_{(3)}^{(6)}$, then the generating scheme (2.15) has order (4, 2) of approximation

$$\Psi = O(|h|^4 + \tau^2) \tag{6}$$

for the solution $u = u(x, t)$ of equation (2.1).

In fact, for any h and τ we have $\psi = O(|h|^4 + \tau^2)$. If $0 \leqslant \sigma_\alpha < 1/2$, then $Q_p u_{\bar{t}}$ and $\sigma_1\sigma_2\sigma_3\Lambda_1\Lambda_2\Lambda_3(u - \check{u})$ are bounded; therefore $\Psi = \psi + O(\tau^2)$. Suppose, for example, that $\sigma_2 \geqslant 0$, $\sigma_3 \geqslant 0$, and $\sigma_1 < 0$, i.e. $(h_1^2/6\tau) > 1$ and, therefore, $|\sigma_1| \leqslant h_1^2/12\tau$. In this case

$$\tau^2 |\sigma_1\sigma_\alpha\Lambda_1\Lambda_\alpha u_{\bar{t}}| \leqslant \frac{\tau h_1^2}{24} |\Lambda_1\Lambda_\alpha u_{\bar{t}}| \leqslant M(\tau^2 + h_1^4), \qquad \alpha = 2, 3,$$

and so on. It is easy to see that $\Psi = \psi + O(|h|^4)$, if all $\sigma_\alpha < 0$. This proves the validity of the estimate (6). We mention only that for $\sigma_\alpha < 0$, $\alpha = 1, 2, 3$, the derivative $\partial^6 u/\partial x_1^2 \partial x_2^2 \partial x_3^2$ must satisfy the Lipschitz condition w.r.t. t, since $\tau^3\sigma_1\sigma_2\sigma_3 \leqslant 12^{-3} h_1^2 h_2^2 h_3^2$, and $h_\alpha^2 > 6\tau$.

Below we shall always assume that the conditions for which the generating scheme (2.15) has maximum order of approximation (6) are satisfied.

2. By analogy with the case $p = 1$ of Section 1 we examine the stability and accuracy by the method of energy inequalities. We introduce the scalar products

$$(y, v) = \sum_{\omega_h} yvH, \qquad (y, v]_\alpha = \sum_{\omega_h^{+\alpha}} yvH, \qquad H = h_1 \ldots h_p,$$

where $\omega_h^{+\alpha} = \omega_h + \gamma_\alpha^+$, and the associated norm

$$\| v \| = \sqrt{(v, v)}, \qquad \| v_{\bar{x}_\alpha} \| = \sqrt{(v_{\bar{x}_\alpha}, v_{\bar{x}_\alpha}]_\alpha}.$$

Lemma 2. Let v be a net function defined on $\bar{\omega}_h$ and equal to zero on the boundary γ of the net $\bar{\omega}_h$ ($v = 0$ for $x \in \gamma$). Then we have the relations

$$\| v \|^2 \leqslant \frac{l_\alpha^2}{4} \| v_{\bar{x}_\alpha} \|^2, \qquad \| v \|^2 \leqslant M_0 I, \qquad I = \sum_{\alpha=1}^{p} \| v_{\bar{x}_\alpha} \|^2,$$

$$M_0 = \frac{1}{4}\left(\sum_{\alpha=1}^{p} \frac{1}{l_\alpha^2}\right)^{-1}, \tag{7}$$

$$(v, \Lambda_\alpha v) = -\| v_{\bar{x}_\alpha} \|^2, \qquad (v, \Lambda_\alpha\Lambda_\beta v) = -(v_{\bar{x}_\alpha}, \Lambda_\beta v_{\bar{x}_\alpha}]_\alpha = \| v_{\bar{x}_\alpha \bar{x}_\beta} \|^2, \quad \alpha \neq \beta. \tag{8}$$

As the proof of the lemma is elementary, we omit it here.

Lemma 3. Let z be a net function defined on $\bar{\omega}_h$ with $z|_\gamma = 0$. Then we have the relation

$$2\,(\Lambda_\alpha\Lambda_\beta\check{z},\ z_{\bar t}) = (\|z_{\bar x_\alpha \bar x_\beta}\|^2)_{\bar t} - \tau\|z_{\bar x_\alpha \bar x_\beta \bar t}\|^2, \quad \alpha\neq\beta, \tag{9}$$

$$h_1^2\,\|z_{\bar x_1\bar x_2\bar x_3}\|^2 \leqslant 4\,\|z_{\bar x_2\bar x_3}\|^2, \tag{10}$$

$$h_1^2 h_2^2\,\|z_{\bar x_1\bar x_2\bar x_3}\|^2 \geqslant 16\,\|z_{\bar x_3}\|^2. \tag{11}$$

The identity (9) follows from (8) and from the formula $2\check{v}v_{\bar t} = (v^2)_{\bar t} - \tau v_{\bar t}^2$. From Lemma 1 we have $h_\alpha^2\,\|v_{\bar x_\alpha}\|^2 \leqslant 4\,\|v\|^2$. Putting $v = z_{\bar x_2\bar x_3}$, we obtain (10). Applying Lemma 1 twice we obtain $h_1^2h_2^2\,\|v_{\bar x_1\bar x_2}\|^2 \leqslant 4^2\,\|v\|^2$, where $v = z_{\bar x_3}$.

3. Let us now derive *a priori* estimates for the solution of equation (1) with zero boundary conditions

$$z = 0 \quad \text{for } x\in\gamma. \tag{12}$$

We rewrite equation (1) in the form

$$\begin{aligned} z_{\bar t} = 0.5\,\Lambda\,(z+\check{z}) - \frac{1}{12}\sum_{\alpha=1}^{p} h_\alpha^2\Lambda_\alpha z_{\bar t} + \tau R_p\check{z} - \tau^2 Q_p z_{\bar t} + \\ + \delta_{p,3}\tau^3\,\sigma_1\sigma_2\sigma_3\Lambda_1\Lambda_2\Lambda_3 z_{\bar t} + \Psi, \qquad \Lambda = \sum_{\alpha=1}^{p}\Lambda_\alpha. \end{aligned} \tag{13}$$

Multiplying (13) scalarly by $2z_{\bar t}$, using Green's formula and remembering that $z\,|_\gamma = 0$, we obtain the basic energy identity

$$\begin{aligned} 2\tau\,\|z_{\bar t}\|^2 + I + 2\tau^3\,(Q_p z_{\bar t},\ z_{\bar t}) + 2\delta_{p,3}\tau^4\sigma_1\sigma_2\sigma_3\,\|z_{\bar x_1\bar x_2\bar x_3\bar t}\|^2 = \\ = \check{I} + \frac{\tau}{6}\sum_{\alpha=1}^{p} h_\alpha^2\,\|z_{\bar x_\alpha\bar t}\|^2 + 2\tau^2\,(R_p\check{z},\ z_{\bar t}) + 2\tau\,(\Psi,\ z_{\bar t}). \end{aligned} \tag{14}$$

From Lemma 3 we have

$$2\tau^2\,(R_p\check{z},\ z_{\bar t}) = \sum_{\alpha=1}^{p}\sum_{\beta>\alpha}(1-\sigma_\alpha-\sigma_\beta)\,[\tau^2\,(\|z_{\bar x_\alpha\bar x_\beta}\|^2)_{\bar t} - \tau^3\,\|z_{\bar x_\alpha\bar x_\beta\bar t}\|^2], \tag{15}$$

$$2\tau^3\,(Q_p z_{\bar t},\ z_{\bar t}) = 2\tau^3\sum_{\alpha=1}^{p}\sum_{\beta>\alpha}\sigma_\alpha\sigma_\beta\,\|z_{\bar x_\alpha\bar x_\beta\bar t}\|^2. \tag{16}$$

Then using the relation

$$1-\sigma_\alpha-\sigma_\beta + 2\sigma_\alpha\sigma_\beta = 0.5\,[1+(1-2\sigma_\alpha)\,(1-2\sigma_\beta)] \geqslant 0.5, \quad \text{since } \sigma_\alpha\leqslant 0.5, \tag{17}$$

we find

$$2\tau^3 (Q_p z_{\bar t}, z_{\bar t}) - 2\tau^2 (R_p \check z, z_{\bar t}) = 0.5\tau^3 \sum_{\alpha=1}^{p} \sum_{\beta>\alpha} [1 + (1-2\sigma_\alpha)(1-2\sigma_\beta)] \| z_{\bar x_\alpha \bar x_\beta \bar t} \|^2 - \tau \sum_{\alpha=1}^{p} \sum_{\beta>\alpha} \frac{h_\alpha^2 + h_\beta^2}{12} (\| z_{\bar x_\alpha \bar x_\beta} \|^2)_{\bar t}. \tag{18}$$

Substituting (18) in (14) we obtain

$$2\tau \| z_{\bar t} \|^2 + I + 0.5\tau^3 \sum_{\alpha=1}^{p} \sum_{\beta=\alpha+1}^{p} [1 + (1-2\sigma_\alpha)(1-2\sigma_\beta)] \| z_{\bar x_\alpha \bar x_\beta \bar t} \|^2 + 2\delta_{p,3} \tau^4 \sigma_1\sigma_2\sigma_3 \| z_{\bar x_1 \bar x_2 \bar x_3 \bar t} \|^2 = \check I + \frac{\tau}{6} \sum_{\alpha=1}^{p} h_\alpha^2 \| z_{\bar x_\alpha \bar t} \|^2 + \tau \sum_{\alpha=1}^{p} \sum_{\beta=\alpha+1}^{p} \frac{h_\alpha^2 + h_\beta^2}{12} (\| z_{\bar x_\alpha \bar x_\beta} \|^2)_{\bar t} + 2\tau (\Psi, z_{\bar t}). \tag{19}$$

Lemma 1 gives

$$\frac{\tau}{6} \sum_{\alpha=1}^{p} h_\alpha^2 \| z_{\bar x_\alpha \bar t} \|^2 \leqslant \frac{2}{3} p\tau \| z_{\bar t} \|^2 \leqslant 2\tau \| z_{\bar t} \|^2 \text{ for } p \leqslant 3. \tag{20}$$

4. Let us examine the cases $p = 2$ and $p = 3$ separately. Let $p = 2$. Using the estimate

$$2\tau (\Psi, z_{\bar t}) \leqslant \tfrac{2}{3} \tau \| z_{\bar t} \|^2 + \tfrac{3}{2} \tau \| \Psi \|^2,$$

from (19) and (20) we obtain the energy inequality

$$I^{j'+1} + 0.5\tau_{j'+1}^3 \| z_{\bar x_1 \bar x_2 \bar t}^{j'+1} \|^2 \leqslant I^{j'} + \frac{h_1^2 + h_2^2}{12} \tau_{j'+1} (\| z_{\bar x_1 \bar x_2}^{j'+1} \|^2)_{\bar t} + \frac{3}{2} \tau_{j'+1} \| \Psi^{j'+1} \|^2.$$

Let us sum w.r.t. $j' = 0, 1, \ldots, j$:

$$I^{j+1} \leqslant I^0 + \frac{h_1^2 + h_2^2}{12} \| z_{\bar x_1 \bar x_2}^{j+1} \|^2 + \frac{3}{2} \left(\overline{\| \Psi^{j+1} \|} \right)^2, \tag{21}$$

where

$$\overline{\| \Psi^{j+1} \|} = \left(\sum_{j'=1}^{j+1} \tau_{j'} \| \Psi^{j'} \|^2 \right)^{1/2}. \tag{22}$$

We now use Lemma 1, which gives

$$(h_1^2 + h_2^2) \| z_{\bar x_1 \bar x_2} \|^2 \leqslant 4I.$$

It follows from this, and from (21) that

$$I^{j+1} \leqslant \tfrac{3}{3} I^0 + \tfrac{9}{4} \left(\overline{\| \Psi^{j+1} \|} \right)^2.$$

Using Lemma 2 we find

$$\| z^{j+1} \| \leqslant \sqrt{\tfrac{3}{2} M_0}\, (\| z_{\bar{x}_1}(x, 0) \| + \| z_{\bar{x}_2}(x, 0) \|) + \tfrac{3}{2}\sqrt{M_0}\, \overline{\| \Psi^{j+1} \|}, \quad (23)$$

where

$$M_0 = \frac{1}{4}\left(\sum_{\alpha=1}^{2} \frac{1}{l_\alpha^2}\right)^{-1} = \frac{l_1^2 l_2^2}{4(l_1^2 + l_2^2)}.$$

This proves the following theorem.

Theorem 3. For $p = 2$ the generating scheme (1) is always stable w.r.t. the initial data and w.r.t. the right-hand side on any sequence of nets $\bar{\Omega}$, so that the solution of equation (1) with boundary conditions $z|_\gamma = 0$ satisfies estimate (23). For $p = 2$ the solution of problem (1)-(2) always satisfies the estimate

$$\| z^{j+1} \| \leqslant \tfrac{3}{2}\sqrt{M_0}\, \overline{\| \Psi^{j+1} \|}. \quad (24)$$

5. Let $p = 3$. In this case the term $2\tau(\Psi, z_{\bar{t}})$ can be estimated in another way (see Section 1):

$$2\tau(\Psi, z_{\bar{t}}) = 2\tau(\Psi, z)_{\bar{t}} - 2\tau(\Psi_{\bar{t}}, \check{z}) \leqslant 2\tau(\Psi, z)_{\bar{t}} + c_0\tau\check{I} + \frac{M_0}{c}\tau\|\Psi_{\bar{t}}\|^2, \quad (25)$$

where c_0 is an arbitrary positive constant, $\Psi(x, 0) = \Psi(x, \tau_1)$. We shall use the obvious estimates:

$$\left.\begin{aligned} &\sum_{\alpha=1}^{p} \sum_{\beta=\alpha+1}^{p} \frac{h_\alpha^2 + h_\beta^2}{12} \| z_{\bar{x}_\alpha \bar{x}_\beta} \|^2 \leqslant \frac{p-1}{3} I, \\ &\tau \sum_{\alpha=1}^{3} \frac{h_\alpha^2}{6} \| z_{\bar{x}_\alpha \bar{t}} \|^2 \leqslant 2\tau \| z_{\bar{t}} \|^2 \qquad \text{(cf. (20))}. \end{aligned}\right\} \quad (26)$$

We rewrite (19) in the form

$$I^{j'+1} + \tau_{j'+1}^3 Q^{j'+1} + 2\tau_{j'+1}^4 \sigma_1\sigma_2\sigma_3 \| z^{j'+1}_{\bar{x}_1\bar{x}_2\bar{x}_3\bar{t}} \|^2 \leqslant (1 + c_0\tau_{j'+1}) I^{j'} + \quad (27)$$
$$+ \sum_{\alpha=1}^{3} \sum_{\beta=\alpha+1}^{3} \frac{h_\alpha^2 + h_\beta^2}{12} \tau_{j'+1} (\| z_{\bar{x}_\alpha \bar{x}_\beta} \|^2)_{\bar{t}}^{j'+1} + \frac{M_0}{c_0}\tau_{j'+1} \| \Psi_{\bar{t}}^{j'+1} \|^2 + 2\tau_{j'+1}(\Psi, z)_{\bar{t}}^{j'+1},$$

where

$$Q = 0.5 \sum_{\alpha=1}^{3} \sum_{\beta=\alpha+1}^{3} [1 + (1 - 2\sigma_\alpha)(1 - 2\sigma_\beta)] \| z_{\bar{x}_\alpha \bar{x}_\beta \bar{t}} \|^2 > 0.$$

The coefficient $\sigma_\alpha = 0.5\,(1 - h_\alpha^2/6\tau)$ can be positive or negative. In particular, we can have $\sigma_\alpha < 0$, which corresponds to the condition $\tau/h_\alpha^2 < {}^1/_6$ or $h_\alpha^2/\tau > 6$. Therefore the product $\sigma_1\sigma_2\sigma_3$ can have any sign and the third term on the left-hand side of (27) cannot be omitted without invalidating the inequality. Let us transform this term. Consider the factor

$$2\sigma_1\sigma_2\sigma_3\tau^4 = \frac{\tau^4}{4} + \frac{h_1^2h_2^2 + h_1^2h_3^2 + h_2^2h_3^2}{144}\tau^2 - D_1^0 - D_2^0 \geqslant - D_1^0 - D_2^0,$$

where

$$D_1^0 = \frac{\tau^3}{24}(h_1^2 + h_2^2 + h_3^2)\,, \qquad D_2^0 = \frac{\tau}{4\cdot 216}h_1^2h_2^2h_3^2.$$

We find an estimate for the expressions

$$D_k = D_k^0\,\| z_{\bar{x}_1\bar{x}_2\bar{x}_3\bar{t}} \|^2, \qquad k = 1,2.$$

Applying Lemma 1 we obtain

$$D_1 \leqslant \frac{\tau^3}{6}\sum_{\alpha=1}^{3}\sum_{\beta=\alpha+1}^{3}\| z_{\bar{x}_\alpha\bar{x}_\beta\bar{t}} \|^2 \leqslant \frac{\tau^3}{3}\,Q,$$

$$D_2 \leqslant \frac{\tau}{4\cdot 216\cdot 3}\{h_1^2h_2^2(h_3^2\,\| z_{\bar{x}_1\bar{x}_2\bar{x}_3\bar{t}} \|^2) + h_1^2h_3^2\,(h_2^2\,\| z_{\bar{x}_1\bar{x}_2\bar{x}_3\bar{t}} \|^2) +$$

$$+ h_2^2h_3^2\,(h_1^2\,\| z_{\bar{x}_1\bar{x}_2\bar{x}_3\bar{t}} \|^2)\} \leqslant \frac{\tau^2\cdot 36}{216\cdot .3}\sum_{\alpha=1}^{3}\sum_{\beta=\alpha+1}^{3}\frac{h_\alpha^2h_\beta^2}{6^2\tau^2}\,\| z_{\bar{x}_\alpha\bar{x}_\beta\bar{t}} \|^2 =$$

$$= \frac{\tau^3}{18}\sum_{\alpha=1}^{3}\sum_{\beta=\alpha+1}^{3}(1 - 2\sigma_\alpha)\,(1 - 2\sigma_\beta)\,\| z_{\bar{x}_\alpha\bar{x}_\beta\bar{t}} \|^2 \leqslant \frac{\tau^3}{9}\,Q.$$

Thus for any $\sigma_1\sigma_2\sigma_3$

$$\tau^3 Q + 2\tau^4\sigma_1\sigma_2\sigma_3\,\| z_{\bar{x}_1\bar{x}_2\bar{x}_3\bar{t}} \|^2 \geqslant \tau^3 Q - D_1 - D_2 \geqslant \tfrac{5}{9}\tau^3\,Q > 0 \qquad (28)$$

and therefore the inequality

$$I^{j+1} \leqslant c_0\sum_{j'=1}^{j+1}\tau_{j'}I^{j'-1} + I^0 + 2\,[(\Psi,\,z)^{j+1} - (\Psi,\,z)^0] + \frac{2}{3}I^{j+1} + \qquad (29)$$
$$+ \frac{M_0}{c_0}\,(\overline{\| \Psi_{\bar{t}}^{j+1} \|})^2,$$

which is obtained from (27), (28) and (29) is always satisfied. Writing $z(x,\ 0) = 0$, using the estimate

$$2\,(\Psi,\,z) \leqslant c_0' I + \frac{M_0}{c_0'}\,\| \Psi \|^2, \qquad c_0' = \text{const.} > 0,$$

and choosing $c_0 = 1/6t_{j+1}$, $c_0' = 1/6$, we find

$$I^{j+1} \leqslant \frac{1}{t_{j+1}} \sum_{j'=2}^{j+1} \tau_{j'} I^{j'-1} + 12M_0 \left[\max_{1\leqslant j'\leqslant j+1} \| \Psi^{j'} \|^2 + t_{j+1} \left(\overline{\| \Psi_{\bar{t}}^{j+1} \|}\right)^2 \right]. \quad (30)$$

Now applying Lemma 4 of [7] we obtain

$$I^{j+1} \leqslant 12M_0 e \left[\max_{1\leqslant j'\leqslant j+1} \| \Psi^{j'} \|^2 + t_{j+1} (\overline{\| \Psi_{\bar{t}}^{j+1} \|})^2 \right]$$

and therefore

$$\| z^{j+1} \| \leqslant M_1 \left(\max_{1\leqslant j'\leqslant j+1} \| \Psi^{j'} \| + \sqrt{t_{j+1}} \overline{\| \Psi_{\bar{t}}^{j+1} \|} \right), \quad (31)$$

where

$$M_0 = \frac{1}{4} \left(\sum_{\alpha=1}^{3} \frac{1}{l_\alpha^2} \right)^{-1}, \qquad M_1 = 2\sqrt{3e}\, M_0.$$

Theorem 4. On any sequence of nets $\bar{\Omega}$ the generating scheme (1) is absolutely stable w.r.t. to the initial data and the right-hand side. The *a priori* estimate (31) applies to the solution of problem (1)-(2) for $p = 3$. The solution of the homogeneous equation (1) ($\Psi = 0$) with boundary conditions $z = 0$ for $x \in \gamma$ satisfies the estimate

$$\frac{1}{\sqrt{M_0}} \| z^{j+1} \| \leqslant \| z_{\bar{x}}^{j+1} \| \leqslant \sqrt{3}\, \| z_{\bar{x}}(x, 0) \|, \qquad \| z_{\bar{x}} \| = \sum_{\alpha=1}^{3} \| z_{\bar{x}_\alpha} \|. \quad (32)$$

The *a priori* estimate (32) follows from (26) and from inequality (29) which gives $I^{j+1} \leqslant 3I^0$. Thus, the scheme (2.13) is absolutely stable w.r.t. the initial data in the norm $\| z_{\bar{x}} \|$.

Note. For the second generating scheme (2.14) we have only succeeded in proving estimate (31) with the additional condition $\sigma_1\sigma_2\sigma_3 \geqslant 0$, for the quasi-uniform net $\bar{\omega}_\tau$ ($|\tau_{\bar{t}}| \leqslant m^*\tau$, (see [7]).

6. After the *a priori* estimates (23) and (31) have been established it is not difficult to show that the generating scheme and, therefore, all the algorithms of Section 3, have fourth order of accuracy w.r.t. $|h|$ and second order accuracy w.r.t. τ.

Theorem 5. Let the conditions for which the scheme (2.15) has maximum order approximation be satisfied: $\| \Psi \| = O(|h|^4 + \tau^2)$ for $p = 2, 3$

and, in addition $\| \Psi_{\bar{t}} \| = O(|h|^4 + \tau^2)$ for $p = 3$. Then the scheme (2.15) converges in the mean on any sequence of nets at a rate $O(|h|^4 + \tau^2)$, so that for any h_α and τ we have the estimates

$$\| y^{j+1} - u^{j+1} \| \leqslant M (|h|^4 + \| \tau^2 \|_{j+1}) \qquad \text{for } p = 2,$$
$$\| y^{j+1} - u^{j+1} \| \leqslant M (|h|^4 + \| \tau \|^2_{0,\, j+1}) \qquad \text{for } p = 3,$$

where $\| \tau \|_{0, j+1} = \max\limits_{1 \leqslant j' \leqslant j+1} \tau_{j'}$, u is a solution of the problem (2.1)-(2.2), y is a solution of the difference problem (2.15)-(2.16) and M are positive constants which do not depend on the choice of nets.

In order to prove the theorem it is sufficient to use the *a priori* estimates (24), (31) and the conditions of the theorem for $\| \Psi \|$ and $\| \Psi_{\bar{t}} \|$.

Note. Theorem 5 still holds for scheme (2.14) if $\sigma_1 \sigma_2 \sigma_3 \geqslant 0$, and the net $\bar{\omega}_\tau$ is quasi-uniform.

5. Three-layer schemes of high-order accuracy

1. We have only succeeded in justifying the two-layer high-order schemes (proving their unconditional stability and convergence) for $p \leqslant 3$. In this section we consider a three-layer scheme (connecting the values of y^{j+1}, y^j and y^{j-1} on three time layers) which, for $p \leqslant 4$, is unconditionally stable and has accuracy $O(|h|^4 + \tau^2)$.

Let us consider the problem

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} \frac{\partial^2 u}{\partial x_\alpha^2}, \qquad u|_\Gamma = \mu, \qquad u(x, 0) = u_0(x) \tag{1}$$

in the cube $0 \leqslant x_\alpha \leqslant 1$, $\alpha = 1, 2, \ldots, p$. Let $\bar{\omega}_h$ be a square net, i.e. $h_\alpha = h_\beta = h =$ const., $\alpha, \beta = 1, 2, \ldots, p$ and let the net $\bar{\omega}_h$ be uniform.

In [1] the initial schemes

$$y_{\overset{\circ}{t}} = \frac{1}{3} \Lambda (y + \check{y} + \check{\check{y}}) - \frac{h^2}{12} y_{\bar{t}t} + \frac{h^2}{6} \sum_{\alpha=1}^{p} \sum_{\beta > \alpha} \Lambda_\alpha \Lambda_\beta \check{y}, \qquad p \leqslant 4, \tag{2}$$

$$y_{\mathring{t}} = \frac{1}{3}\Lambda\,(y + \check{y} + \check{\check{y}}) - \frac{h^2}{12}\Lambda y_{\mathring{t}} + \frac{h^2}{6}\sum_{\alpha=1}^{p}\sum_{\beta>\alpha}\Lambda_\alpha\Lambda_\beta\check{y}, \qquad p \leqslant 3, \quad (3)$$

were proposed for this problem, where

$$y = y^{j+1}, \quad \check{y} = y^{j}, \quad \check{\check{y}} = y^{j-1}, \quad y_{\mathring{t}} = (y - \check{\check{y}})/2\tau = 0.5\,(y_{\bar{t}} + \check{y}_{\bar{t}}). \quad (4)$$

It was shown that the initial scheme (3) has the necessary accuracy $\| y - n \| = O\,(h^4 + \tau^2)$ with the supplementary condition

$$\gamma \geqslant \gamma_0 = \text{const.} > 0.$$

The alternating direction algorithm for the scheme (2) has the form

$$\left.\begin{aligned}
&(E - A_1)\,y_{(1)} = \left\{\frac{1}{6\gamma}E + \frac{\tau}{3}(\Lambda - 2\Lambda_1) + \frac{\tau h^2}{3}\sum_{\alpha=1}^{p}\sum_{\beta>\alpha}\Lambda_\alpha\Lambda_\beta\right\}\check{y} + \\
&\qquad\qquad + \left[\left(1 - \frac{1}{6\gamma}\right)E + \frac{2\tau}{3}\Lambda\right]\check{\check{y}}, \\
&(E - A_\alpha)\,y_{(\alpha)} = y_{(\alpha-1)} - A_{\alpha-1}\check{y}, \quad \alpha > 1, \qquad y_{(p)} = y = y^{j+1}, \\
&A_1 = \frac{2}{3}\tau\Lambda_1 - \frac{1}{6\gamma}E, \qquad A_\alpha = \frac{2\tau}{3}\Lambda_\alpha, \qquad \alpha > 1.
\end{aligned}\right\} \quad (5)$$

The generating scheme was not given, the authors undertaking to prove its convergence later. The question of the boundary conditions for $\alpha < p$ (see [5]) was not discussed.

2. Our aim is to construct a three-layer scheme for equation (1) which is unconditionally stable and convergent at a rate $O\,(h^4 + \tau^2)$ (for any γ) for $p \leqslant 4$. By analogy with Section 2 we introduce the parameter

$$\sigma = \frac{0.5}{1 + 1/12\gamma}. \quad (6)$$

If we put $\sigma = 0.5$ formally in the scheme described below it becomes a two-layer scheme of $O\,(h^2 + \tau^2)$.

Thus, let us consider problem (1) on the same net as in [1]. We put $h_\alpha = h$ and replace $\sum_{\alpha=1}^{p}\Lambda_\alpha y_{\bar{t}} = \Lambda y_{\bar{t}}$ by the expression $y_{\bar{t}\bar{t}}$ in the scheme (2.3). Then we obtain the following initial scheme*:

* We can also consider the scheme (7) as a two-layer scheme if we assume that $\check{y}_{\bar{t}}$ is given on the previous layer (for $t = t_j$) as well as $\check{y}$.

$$y_{\bar{t}} = 0.5\,\Lambda\,(y + \check{y}) - \frac{h^2}{12} y_{t\bar{t}} + \frac{h^2}{6} \sum_{\alpha=1}^{p} \sum_{\beta>\alpha} \Lambda_\alpha \Lambda_\beta \check{y}. \tag{7}$$

It is easy to see that it has approximation error $O\,(h^4 + \tau^2)$.

We must fix initial data for the scheme (7) not only for $t = 0$ but also for $t = \tau$. To find $y(x,\ \tau)$ we need a two-layer scheme of accuracy $O\,(h^4 + \tau^2)$ (see Paragraph 4). We shall start from (7). Using the relation $\left(\frac{\partial^2 u}{\partial t^2}\right)^{j+1/2} = \frac{2}{\tau}(u_{\bar{t}} - \check{u}_{\bar{t}}) + O\,(\tau)$, we replace $\frac{\partial^2 u}{\partial t^2}\,(x,\,\tau)$ by the expression

$$\frac{1}{\tau}\,y_{\bar{t}} - \frac{1}{\tau}\frac{\partial u}{\partial t}(x,\,0) + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}(x,\,0) = \frac{1}{\tau}\,y_{\bar{t}} - \frac{1}{\tau} L u_0\,(x) + \frac{1}{2}\,LL\,u_0\,(x).$$

As a result we obtain the initial scheme

$$\begin{gathered} y_{\bar{t}} = 0.5\,\Lambda\,(y + \check{y}) - \frac{h^2}{12\tau} y_{\bar{t}} + \varphi, \qquad y\,|_\gamma = \mu,\ y\,(x,\,0) = u_0\,(x), \qquad t = \tau, \\ \varphi = \frac{h^2}{12}\left[\frac{1}{\tau} L u_0 - \frac{1}{2} LL u_0 + 2 \sum_{\alpha=1}^{p} \sum_{\beta=\alpha+1}^{p} L_\alpha L_\beta u_0\right] \end{gathered} \tag{8}$$

for $y(x,\ \tau)$.

3. We rewrite (7) and (8) in the form

$$(E - \sigma\tau\Lambda)\,y_{\bar{t}} = \Phi\,[\check{y}], \qquad y\,|_\gamma = \mu, \qquad y\,(x,\,0) = u_0\,(x), \tag{9}$$

$$\left.\begin{aligned} \Phi\,[\check{y}] &= F\,[\check{y}] = 2\sigma\Lambda\check{y} + (1 - 2\sigma)\,\check{y}_{\bar{t}} + \\ &\quad + 2\,(1 - 2\sigma)\,\tau \sum_{\alpha=1}^{p} \sum_{\beta>\alpha} \Lambda_\alpha \Lambda_\beta \check{y}, \qquad t > \tau, \\ \Phi\,[\check{y}] &= F_1\,[u_0] = 2\sigma\,[\Lambda u_0 + \varphi] \qquad \text{for } t = \tau. \end{aligned}\right\} \tag{10}$$

After substituting for the operator $E - \sigma\tau\Lambda$ the product of one-dimensional operators $A_1,\ \ldots,\ A_p = 1$ we obtain the generating scheme

$$A y_{\bar{t}} = \prod_{\alpha=1}^{p} A_\alpha y_{\bar{t}} = \Phi\,[\check{y}], \qquad y\,|_\gamma = \mu, \qquad y\,(x,\,0) = u_0\,(x), \tag{11}$$

where

$$A_\alpha = E - \sigma\tau\Lambda_\alpha. \tag{12}$$

We obtain at once the alternating direction computing algorithm

$$A_1 v_{(1)} = \Phi\,[\check{y}],\ A_\alpha v_{(\alpha)} = v_{(\alpha-1)} \text{ for } \alpha > 1, \qquad y = \check{y} + \tau v_{(p)}, \qquad t \geqslant \tau, \tag{13}$$

$$v_{(\alpha)} = A_{\alpha+1}, \ldots A_p\,(\mu_\alpha)_{\bar{t}}^{j+1} \text{ for } x \in \gamma_\alpha \qquad (x_\alpha = 0,\ \ x_\alpha = 1) \tag{14}$$

where $\Phi\,[\check{y}]$ is given by the formulae (10).

4. Let u be a solution of problem (1), y a solution of problem (11). For the error $z = y - u$ we obtain

$$\left.\begin{aligned} &A z_{\bar{t}} = F\,[\check{z}] + 2\sigma\Psi \text{ for } t > \tau, \qquad A z_{\bar{t}} = 2\sigma\psi \text{ for } t = \tau, \\ &z\,|_\gamma = 0 \quad \text{for} \quad t \in \bar{\omega}_\tau; \qquad z\,(x, 0) = 0, \qquad x \in \bar{\omega}_h, \end{aligned}\right\} \tag{15}$$

where Ψ and ψ are the approximation error of the scheme for $t > \tau$ and, correspondingly, for $t = \tau$ for the solution $u = u(x,\ t)$ of equation (1). It is clear from the construction of the scheme that

$$\Psi = O\,(h^4 + \tau^2), \qquad \psi = O\,(h^4 + \tau^2),$$

if the solution $u = u(x,\ t)$ of equation (1) is sufficiently smooth.

Let us go on to derive *a priori* estimates from which the stability and convergence of our scheme will follow. The argument is similar to that of Section 4. It must be borne in mind here that $\sigma > 0$ always. Multiplying (15) scalarly by $\tau z_{\bar{t}}$ and using the relation

$$\tau\,(A z_{\bar{t}},\ z_{\bar{t}}) = \tau\,\|\,z_{\bar{t}}\,\|^2 + \sigma\tau^2 \sum_{\alpha=1}^{p} \|\,z_{\bar{x}_\alpha \bar{t}}\,\|^2 + \sigma^2\tau^3 \sum_{\alpha=1}^{p} \sum_{\beta>\alpha} \|\,z_{\bar{x}_\alpha \bar{x}_\beta \bar{t}}\,\|^2 + \cdots$$
$$\cdots + \sigma^p \tau^{p+1}\,\|\,z_{\bar{x}_1\,\bar{x}_2 \ldots \bar{x}_p \bar{t}}\,\|^2,$$

$$\tau\,[\,\|\,z_{\bar{t}}\,\|^2 - (1 - 2\sigma)\,(\check{z}_{\bar{t}},\ z_{\bar{t}})] = 2\sigma\tau\,\|\,z_{\bar{t}}\,\|^2 + (0.5 - \sigma)\,\tau^2\,(\|\,z_{\bar{t}}\,\|^2)_{\bar{t}} +$$
$$+ (0.5 - \sigma)\,\tau^3\,\|\,z_{\bar{t}t}\,\|^2,$$

$$2\sigma\tau\,(\Lambda\check{z},\ z_{\bar{t}}) = -\,\sigma\tau I_{\bar{t}} + \sigma\tau^2 \sum_{\alpha=1}^{p} \|\,z_{\bar{x}_\alpha \bar{t}}\,\|^2, \qquad I = \sum_{\alpha=1}^{p} \|\,z_{\bar{x}_\alpha}\,\|^2,$$
$$2\tau\,(z_{\bar{t}t},\ z_{\bar{t}}) = \tau\,(\|\,z_{\bar{t}}\,\|^2)_{\bar{t}} + \tau^2\,\|\,z_{\bar{t}t}\,\|^2, \qquad 2\tau\,(\Psi,\ z_{\bar{t}}) \leqslant \tau\,\|\,z_{\bar{t}}\,\|^2 + \tau\,\|\,\Psi\,\|^2,$$

we obtain an energy inequality and solving this, after the usual reasoning, we arrive at the following estimates:

$$\tau\,\|\,z_{\bar{t}}\,(x,\ \tau)\,\|^2 + \frac{h^2}{12}\,\|\,z_{\bar{t}}\,(x,\ \tau)\,\|^2 + \tau^2 \sum_{\alpha=1}^{p} \|z_{\bar{x}_\alpha \bar{t}}\,(x,\ \tau)\|^2 \leqslant \|\,\psi\,\|^2, \tag{16}$$

$$\sum_{j'=2}^{j+1} \tau \| z_{\bar{t}}^{j'} \|^2 + \frac{h^2}{12} \| z_{\bar{t}}^{j+1} \|^2 + I^{j+1} \leqslant I(\tau) + \frac{h^2}{12} \| z_{\bar{t}}(x, \tau) \|^2 + \\ + \frac{h^2}{6} \sum_{\alpha=1}^{p} \sum_{\beta>\alpha} \| z_{\bar{x}_\alpha \bar{x}_\beta}^{j+1} \|^2 + (\overline{\| \Psi^{j+1} \|})^2, \tag{17}$$

$$\tau^2 \sum_{\alpha=1}^{p} \| z_{\bar{x}_\alpha \bar{t}}(x, \tau) \|^2 = I(\tau) \quad \text{for} \qquad z(x, 0) = 0.$$

It follows from (16), (17), (4.26) and the condition $z(x, 0) = 0$ that

$$\sum_{j'=1}^{j+1} \tau \| z_{\bar{t}}^{j'} \|^2 + \frac{h^2}{12} \| z_{\bar{t}}^{j+1} \|^2 + \frac{4-p}{3} I^{j+1} \leqslant \| \psi \|^2 + (\overline{\| \Psi^{j+1} \|})^2. \tag{18}$$

For the estimate of $\| z^{j+1} \|$ for $p = 4$ we need the following obvious lemma.

Lemma 4. If $z(x, 0) = 0$, then

$$\| z^{j+1} \|^2 \leqslant t_{j+1} \sum_{j'=1}^{j+1} \tau \| z_{\bar{t}}^{j'} \|^2. \tag{19}$$

5. Using (17), (18) and Lemma 3 we obtain the following theorem.

Theorem 6. The generating scheme (15) is absolutely stable for $p \leqslant 4$, so that for any h and τ the solution of the problem (15) satisfies the inequality

$$\| z^{j+1} \| \leqslant \sqrt{t_{j+1}} \, (\| \psi \| + \overline{\| \Psi^{j+1} \|}). \tag{20}$$

Theorem 7. If the conditions for which scheme (11) has maximum order of approximation for the solution $u = u(x, t)$ of the problem (1) are satisfied, i.e.

$$\| \Psi \| = O(h^4 + \tau^2), \qquad \| \psi \| = O(h^4 + \tau^2), \tag{21}$$

then it converges at a rate $O(h^4 + \tau^2)$:

$$\| y - u \| = O(h^4 + \tau^2) \tag{22}$$

for any values of γ for $p \leqslant 4$.

The proof of Theorem 7 follows immediately from Theorem 6 and conditions (21).

Note. In [1] an estimate of the form

$$\| z^{j+1} \| \leqslant C(\gamma) \| z(x, \tau) \|, \tag{23}$$

where $C(\gamma)$ is a constant which does not depend on $\gamma = \tau/h^2$, was obtained for the initial scheme (2).

6. For simplicity and convenience in our comparison with [1] we have considered the scheme on the square net $\bar{\omega}_h$. If $\bar{\omega}_h$ is the net of Section 2, so that $h_\alpha \neq h_\beta$, instead of (7) we have the scheme

$$y_{\bar{t}} = 0.5\,\Lambda\,(y + \check{y}) - \frac{|h|^2}{12} y_{\bar{t}t} + \frac{1}{12} \sum_{\alpha=1}^{p} (|h|^2 - h_\alpha^2)\,\Lambda_\alpha y_{\bar{t}} + \tag{24}$$

$$+ \sum_{\alpha=1}^{p} \sum_{\beta>\alpha} \frac{h_\alpha^2 + h_\beta^2}{12} \Lambda_\alpha \Lambda_\beta \check{y}, \qquad |h|^2 = h_1^2 + \cdots + h_p^2.$$

It is not difficult to construct the generating scheme

$$A_1 \ldots A_p y_{\bar{t}} = F[\check{y}], \qquad A_\alpha = E - \tau\sigma_\alpha\Lambda_\alpha,$$

$$\sigma_\alpha = \frac{1}{2}\left(1 + \frac{|h|^2 - h_\alpha^2}{6\tau}\right) \Big/ \left(1 + \frac{|h|^2}{6\tau}\right), \tag{25}$$

and we can see that Theorems 6 and 7 remain valid in this case also, for $p \leqslant 4$.

The scheme for a non-homogeneous equation (1) is written by analogy with Section 2. The problem of finding schemes of accuracy $O(|h|^4 + \tau^2)$ which are absolutely stable for $p > 4$ is of interest.

6. A scheme of high-order accuracy for an equation with mixed derivatives

1. Economical schemes of accuracy $O(h^2 + \tau)$ were described in [14], [6], [11] for the parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{\alpha,\beta=1}^{p} a_{\alpha\beta} \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta}, \qquad \sum_{\alpha,\beta=1}^{p} a_{\alpha\beta}\xi_\alpha\xi_\beta \geqslant c_1 \sum_{\alpha=1}^{p} \xi_\alpha^2, \qquad c_1 = \text{const.} > 0. \tag{1}$$

Let us show that we can construct a scheme of accuracy $O(h^4 + \tau^2)$ for the case $p = 2$, when $a_{\alpha\beta}$ = const.

Without loss of generality we can take $a_{11} = a_{22} = 1$, $a_{12} = a_{21}$, so that

$$|a_{12}| \leqslant 1 - c_1. \tag{2}$$

Thus in the region $0 \leqslant x_\alpha \leqslant l_\alpha$ $(\alpha = 1, 2)$, $0 \leqslant t \leqslant T$ we consider the problem

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t} = Lu, \quad u|_\Gamma = \mu(x, t), \quad u(x, 0) = u_0(x), \\ Lu = (L_1 + L_2 + 2a_{12}L_{12})\, u, \quad L_\alpha u = \dfrac{\partial^2 u}{\partial x_\alpha^2}, \quad \alpha = 1, 2; \quad L_{12}u = \dfrac{\partial^2 u}{\partial x_1 \partial x_2}. \end{array}\right\} \tag{3}$$

2. Let $\bar{\omega}_h$ be a square net with step h, $\bar{\omega}_\tau$ a uniform net with step τ. To approximate to $L_{12}u$ we use two difference operators:

$$\Lambda_{12}^- u = 0.5\,(u_{\bar{x}_1 x_2} + u_{x_1 \bar{x}_2}), \qquad \Lambda_{12}^+ u = 0.5\,(u_{\bar{x}_1 \bar{x}_2} + u_{x_1 x_2}).$$

After calculation we have

$$\Lambda_{12}^- u = L_2 u - \frac{h^2}{4} L_{12}^2 u + \frac{h^2}{6} L_{12}(L_1 + L_2) u + O(h^4),$$

$$\Lambda_{12}^+ u = L_{12} + \frac{h^2}{4} L_{12}^2 u + \frac{h^2}{6} L_{12}(L_1 + L_2)\, u + O(h^4).$$

Putting

$$\Lambda y = (\Lambda_1 + \Lambda_2 + 2a_{12}\Lambda_{12}^{\mp})\, y, \qquad \Lambda_\alpha y = y_{\bar{x}_\alpha x_\alpha},$$

we find

$$\Lambda u = Lu + \frac{h^2}{12}[L_1^2 + L_2^2 + 4a_{12}L_{12}(L_1 + L_2) \mp 6a_{12}L_{12}^2]\, u + O(h^4).$$

Using equation (2), after a number of calculations we obtain

$$0.5\,(\Lambda_1 + \Lambda_2)\,(u + \check{u}) + 2a_{12}\Lambda_{12}^{\mp}\check{u} - \frac{h^2}{12}u_{\bar{t}t} + \frac{h^2}{6}(1 + 2a_{12}^2 \pm 3a_{12})\,\Lambda_1\Lambda_2\check{u} = \\ = L\check{u} + O(h^4 + \tau^2).$$

The operator $\Lambda_{12}^{\mp}$ is chosen depending on the sign of a_{12}:

$$\begin{array}{ll} \Lambda_{12} = \Lambda_{12}^-, & \text{if } a_{12} < 0, \\ \Lambda_{12} = \Lambda_{12}^+, & \text{if } a_{12} > 0. \end{array} \tag{5}$$

3. Now let us write down the initial scheme of order (4,2) of approximation:

$$y_{\overset{\circ}{t}} = 0.5\,(\Lambda_1 + \Lambda_2)\,(y + \check{\check{y}}) + 2a_{12}\Lambda_{12}\check{y} - \frac{h^2}{12}\,y_{\bar{t}\bar{t}} + \frac{h^2}{6}\,b\Lambda_1\Lambda_2\check{y}, \tag{6}$$

where

$$b = 1 + 2a_{12}^2 - 3\,|a_{12}|, \qquad y_{\overset{\circ}{t}} = (y - \check{\check{y}})/2\tau = 0.5\,(y_{\bar{t}} + \check{y}_{\bar{t}}). \tag{7}$$

The generating scheme will clearly have the form

$$\left.\begin{aligned} &Ay_{\bar{t}} = A_1A_2y_{\bar{t}} = F\,[\check{y}, \check{\check{y}}], \qquad \dot{y}\,|_\gamma = \mu \quad \text{for } t > \tau, \\ &Ay_{\bar{t}} = F_1\,[u_0], \qquad y\,|_\gamma = \mu\,(x_1, \tau), \qquad y\,(x, 0) = u_0\,(x) \quad \text{for } t = \tau, \end{aligned}\right\} \tag{8}$$

where

$$A_\alpha = E - \sigma\tau\Lambda_\alpha, \qquad \sigma = 1\bigg/\left(1 + \frac{h^2}{6\tau}\right), \tag{9}$$

$$F\,[\check{y}, \check{\check{y}}] = (1 - 2\sigma)\,\check{y}_{\bar{t}} + \sigma\Lambda\,(\check{y} + \check{\check{y}}) + 4\sigma a_{12}\Lambda_{12}\check{y} + 2\,(1 - \sigma)\,\tau b\Lambda_1\Lambda_2\check{y}. \tag{10}$$

We shall not write out the expression for $F_1\,[u_0]$ (see Section 5).

The alternating direction algorithm is written by analogy with Section 5. Let u be a solution of problem (3) and y a solution of problem (8); for their difference we have

$$\left.\begin{aligned} Az_{\bar{t}} = F\,[\check{z}, \check{\check{z}}] + 2\sigma\Psi, \quad t > \tau; \quad Az_{\bar{t}} = 2\sigma\psi, \quad t = \tau, \\ z\,|_\gamma = 0, \qquad z\,(x, 0) = 0, \end{aligned}\right\} \tag{11}$$

where $\Psi = O\,(h^4 + \tau^2)$, $\psi = O\,(h^4 + \tau^2)$, if $u = u\,(x, t) \in C_3^{(6)}$.

4. Multiplying (11) scalarly by $2\,(z_{\bar{t}} + \check{z}_{\bar{t}})$ we obtain the energy identity

$$\tau\,\|\,z_{\bar{t}} + \check{z}_{\bar{t}}\|^2 + \tau\,(I + \check{I})_{\bar{t}} + 0.5\,\sigma\tau^4\,(\|\,z_{\bar{x}_1\bar{x}_2\bar{t}}\|^2)_{\bar{t}} + 0.5\,\sigma\tau^3\,\|\,z_{\bar{x}_1\bar{x}_2\bar{t}} + \check{z}_{\bar{x}_1\bar{x}_2\bar{t}}\|^2 -$$

$$- 4a_{12}\,(\Lambda_{12}\check{z}, z - \check{\check{z}}) = -\frac{h^2}{6}\,\tau\,(\|\,z_{\bar{t}}\|^2)_{\bar{t}} + \frac{h^2}{3}\,b\tau\,(z_{\bar{x}_1\bar{x}_2}, \check{z}_{\bar{x}_1\bar{x}_2})_{\bar{t}} + 2\tau\,(\Psi, z_{\bar{t}} + \check{z}_{\bar{t}}).$$

It is not difficult to see that

$$(\Lambda_{12}^-\check{z}, z - \check{\check{z}}) = -0.5\,\tau\,(Q_{12}^-)_{\bar{t}}, \quad Q_{12}^- = (\check{z}_{\bar{x}_1}, z_{\bar{x}_2}) + (z_{\bar{x}_1}, \check{z}_{\bar{x}_2}),$$

$$(\Lambda_{12}^+\check{z}, z - \check{\check{z}}) = -0.5\,\tau\,(Q_{12}^+)_{\bar{t}}, \quad Q_{12}^+ = (\check{z}_{x_1}, z_{\bar{x}_2}) + (z_{x_1}, \check{z}_{\bar{x}_2}).$$

Following Section 5, we find

$$\frac{h^2}{6}\|z_{\bar t}^{j+1}\|^2+I^{j+1}+I^j+0.5\sigma\tau^3\|z_{\bar x_1\bar x_2\bar t}^{j+1}\|^2+2a_{12}Q_{12}(t_{j+1})\leqslant\frac{h^2}{6}\|z_{\bar t}(x,\tau)\|^2+ \\ +I(\tau)+I(0)+\left(\overline{\|\Psi^{j+1}\|}\right)^2+\frac{h^2}{3}b\left(z_{\bar x_1\bar x_2}^{j+1},z_{\bar x_1\bar x_2}^{j}\right)+0.5\sigma\tau^3\|z_{\bar x_1\bar x_2\bar t}(x,\tau)\|^2. \tag{12}$$

We use the identity $(v,\check v)=0.5(\|v\|^2+\|\check v\|^2)-0.5\tau^2\|v_{\bar t}\|^2$, $v=z_{\bar x_1\bar x_2}$. Let $b>0$. Then the term $-0.5\,\tau^2\frac{h^2}{3}b\|z_{\bar x_1\bar x_2\bar t}^{j+1}\|^2$ can be ignored, and Lemma 1 can be used to estimate $h^2\|z_{\bar x_1\bar x_2}\|^2$. As a result on the left we obtain the expression $(1-\frac{1}{3}b)$ $(I^{j+1}+I^j)=[|a_{12}|+\frac{2}{3}(1-a_{12}^2)]$ $(I^{j+1}+I^j)$ on the left. For definiteness let us consider the case $\Lambda_{12}=\Lambda_{12}^-$, i.e. $a_{12}<0$. Writing $\xi_\alpha=z_{\bar x_\alpha}^{j+1}$, $\check\xi_\alpha=z_{\bar x_\alpha}^{j}$, we find $[|a_{12}|+\frac{2}{3}(1-a_{12}^2)](\xi_1^2+\check\xi_2^2)+2a_{12}\xi_1\check\xi_2\geqslant\frac{2}{3}(1-a_{12}^2)(\xi_1^2+\check\xi_2^2)\geqslant\frac{2}{3}c_1^2(\xi_1^2+\check\xi_2^2)$, since $1-a_{12}^2\geqslant c_1^2$.

As a result, inequality (12) takes the form

$$\frac{2}{3}c_1^2(I^{j+1}+I^j)\leqslant I(\tau)+\frac{h^2}{6}\|z_{\bar t}(x,\tau)\|^2+\left(\overline{\|\Psi^{j+1}\|}\right)^2. \tag{13}$$

By analogy with Section 5 we find

$$0.5\,\sigma\tau^3\|z_{\bar x_1\bar x_2\bar t}(x,\tau)\|^2+I(\tau)+\frac{h^2}{6}\|z_{\bar t}(x,\tau)\|^2\leqslant\|\psi\|^2 \tag{14}$$

and therefore

$$I^{j+1}+I^j\leqslant\frac{3}{2c_1^2}[\|\psi\|^2+(\overline{\|\Psi^{j+1}\|})^2]\quad\text{for any }\gamma=\frac{\tau}{h^2}. \tag{15}$$

Since $b>0$ for $|a_{12}|<0.5$ estimate (15) is valid for $c_1=0.5$.

Let $b<0$. Then we can ignore the term

$$0.5\frac{h^2}{3}b(\|z_{\bar x_1\bar x_2}^{j+1}\|^2+\|z_{\bar x_1\bar x_2}^{j}\|^2),$$

in (12) and take the term with $\|z_{\bar x_1\bar x_2\bar t}^{j+1}\|^2$ to the left-hand side. Then on the left we obtain the expression

$$\left(0.5\sigma-\frac{h^2}{6\tau}|b|\right)\tau^3\|z_{\bar x_1\bar x_2\bar t}\|^2.$$

The coefficient $0.5\sigma - h^2|b|/6\tau \geqslant 0$, if the condition

$$\gamma \geqslant \gamma_0 = \frac{1}{6}|b|\left[1+\sqrt{1+\frac{2}{|b|}}\right] \quad \text{for} \quad b = 1 + 2a_{12}^2 - 3\,|a_{12}| < 0 \tag{16}$$

(i.e. for $|a_{12}| > 0.5$) holds.

Since $\max|b| = \frac{1}{8}$, $\max \gamma_0 = [1+\sqrt{17}]/48$. Using (16) we again arrive at (15).

5. We have thus proved the following theorem.

Theorem 8. The solution of problem (11) satisfies the *a priori* estimate

$$\|z^{j+1}\| \leqslant M\,(\|\psi\| + \overline{\|\Psi^{j+1}\|}), \qquad M = \frac{\sqrt{6}}{2c_1}M_0, \tag{17}$$

if condition (16) is satisfied. If $|a_{12}| \leqslant 0.5$ then the estimate (17) is true for any $\gamma = \tau/h^2$.

Theorem 9. If condition (16) holds and

$$\|\psi\| = O\,(h^4 + \tau^2), \qquad \|\Psi\| = O\,(h^4 + \tau^2), \tag{18}$$

then the scheme (8) converges at a rate $O\,(h^4 + \tau^2)$, so that

$$\|y - u\| \leqslant M\,(h^4 + \tau^2), \tag{19}$$

where M is a positive constant which does not depend on h and τ.

Theorem 9 follows from (17) and (18).

Note 1. If the new variables $x'_\alpha = x_\alpha/\sqrt{a_{\alpha\alpha}}$, are introduced, in equation (1) we obtain $a'_{\alpha\alpha} = 1$, $a'_{\alpha\beta} = a_{\alpha\beta}/\sqrt{a_{\alpha\alpha}a_{\beta\beta}}$. The square net $\bar{\omega}_h$ which we have used corresponds to the variable x'_α. Since $\Delta x'_\alpha = \Delta x_\alpha/\sqrt{a_{\alpha\alpha}}$, in the old variables the steps h_α must satisfy the condition $h_\alpha/\sqrt{a_{\alpha\alpha}} = h$ = const. All the results of this section still apply if the steps h_α are so chosen that

$$\frac{h_\alpha^2}{a_{\alpha\alpha}} = \frac{h_\beta^2}{a_{\beta\beta}}\left(1 + O\left(h_\beta^2\right)\right).$$

Note 2. If $c_1 = 0$, i.e. $|a_{12}| \leqslant 1$ then besides (18) and (19) we obtain the estimates

$$\| z^{j+1} + z^{j} \| \leqslant \sqrt{t_{j+1}} (\sqrt{2} \| \psi \| + 2 \overline{\| \Psi^{j+1} \|}). \tag{18'}$$

$$\| (y^{j+1} + y^{j}) - (u^{j+1} + u^{j}) \| \leqslant M (h^4 + \tau^2). \tag{19'}$$

When deriving (18') in Paragraph 4 we must use the estimate

$$2\tau (\Psi, z_{\bar{t}} + \check{z}_{\bar{t}}) \leqslant 0.5\tau \| z_{\bar{t}} + \check{z}_{\bar{t}} \|^2 + 2\tau \| \Psi \|^2$$

for $2\tau(\Psi, z_{\bar{t}} + \check{z}_{\bar{t}})$.

7. Schemes of high-order accuracy for equations with variable coefficients

1. Let us begin by constructing a scheme of order (4,2) for the heat conduction equation for one space variable. Let it be required to solve the problem

$$\frac{\partial u}{\partial t} = Lu + f(x, t), \quad Lu = \frac{\partial}{\partial x}\left(k(x, t)\frac{\partial u}{\partial x}\right), \qquad 0 < x < 1, \quad 0 < t \leqslant T, \tag{1}$$

$$\begin{gathered} u(0, t) = u_1(t), \qquad u(1, t) = u_2(t), \quad 0 \leqslant t \leqslant T; \\ u(x, 0) = u_0(x), \quad k \geqslant c_1 > 0 \end{gathered} \tag{2}$$

in $\bar{D} = (0 \leqslant x \leqslant 1, \ 0 \leqslant t \leqslant T)$.

We consider the homogeneous difference scheme (see [13])

$$\Lambda y = (a y_{\bar{x}})_x, \quad a = a(x, t) = 1/A[p(x + sh, t)], \quad -1 \leqslant s \leqslant 0, \quad p = \frac{1}{k}, \tag{3}$$

approximating to the operator Lu. Here $A[\mu(s)]$ is a linear pattern functional satisfying the conditions

$$A[1] = 1, \quad A[s] = -0.5, \qquad A[s^2] = \tfrac{1}{3}, \quad A[f] \geqslant 0 \quad \text{for } f \geqslant 0. \tag{4}$$

The scheme (3) has second order approximation w.r.t. h. After a number of calculations we find

$$\Lambda u = (a u_{\bar{x}})_x = Lu + \frac{h^2}{12} L(pLu) + O(h^4), \quad p(x, t) = \frac{1}{k(x, t)}. \tag{5}$$

If $u = u(x, t)$ is a solution of equation (1), then we can express Lu

from (1): $Lu = \partial u/\partial t - f$ and substitute in (5):

$$\Lambda u = Lu + \frac{h^2}{12} L\left(p\frac{\partial u}{\partial t} - pf\right) + O(h^4).$$

It follows that the scheme (for the notation see Section 1)

$$y_{\bar{t}} = 0.5\Lambda\,(y + \check{y}) - \frac{h^2}{12}\Lambda\,(py_{\bar{t}}) + \varphi;\quad y_0 = u_1,\quad y_N = u_2;\ y\,(x, 0) = u_0(x), \tag{6}$$

where

$$\varphi = \left[f + \frac{h^2}{12}\Lambda\,(pf)\right]^{j+1/2}.\quad \Lambda y = (a\,(x, t_{j+1/2})\,y_{\bar{x}})_x, \tag{7}$$

has fourth order approximation w.r.t. h and second order approximation w.r.t. τ for the solution $u = u(x, t)$ of equation (1).

We can write equation (6) in the form

$$y_{\bar{t}} = \Lambda\sigma y + \Lambda\,(1 - \sigma)\,y + \varphi,\qquad \sigma = \frac{1}{2}\left(1 - \frac{h^2}{6\tau}p\right). \tag{8}$$

To find y on the new row $t = t_{j+1}$ we obtain the problem:

$$A_i y_{i-1} - C_i y_i + B_i y_{i+1} = -F_i,\qquad y_0 = u_1,\ y_N = u_2 \tag{9}$$

where

$$A_i = a_i\gamma\sigma_{i-1} = 0.5a_i\left(\gamma - \frac{p_{i-1}}{6}\right),\qquad B_i = 0.5a_{i+1}\left(\gamma - \frac{p_{i+1}}{6}\right),$$
$$C_i = 1 + 0.5\,(a_i + a_{i+1})\left(\gamma - \frac{p_i}{6}\right),\qquad \gamma = \tau/h^2.$$

It is clear from this that $C_i = A_i + B_i + D_i$, $D_i = 1 - \frac{h}{12}(a_{i+1}p_{x,i} + a_i p_{\bar{x},i}) > 0$, if $h \leqslant h_0$ is sufficiently small, or, more precisely $h\,|p'/p| \leqslant c^* < 6$. We can use successive substitution formulae for the solution of problem (9).

2. For the error $z = y - u$ where u is the solution of problem (1) and y the solution of problem (6) we obtain the conditions

$$z_{\bar{t}} = 0.5\Lambda\,(z + \check{z}) - \frac{h^2}{12}\Lambda\,(pz_{\bar{t}}) + \psi,\qquad z_0 = z_N = 0, \tag{10}$$

$$z\,(x, 0) = 0, \tag{11}$$

where ψ is the approximation error of the scheme (6):

$$\psi = O\,(h^4 + \tau^2). \tag{12}$$

We investigate the stability of scheme (10). By analogy with Section 1 we write down the basic energy identity

$$\tau \,\| z_{\bar t} \|^2 + 0.5\,(a,\ z_{\bar x}^2] = 0.5\,(a,\ \check z_{\bar x}^2] + \frac{h^2}{12}\tau\,((p z_{\bar t})_{\bar x},\ a z_{\bar x \bar t}] + \tau\,(\psi,\ z_{\bar t}). \tag{13}$$

We transform the sum

$$((p z_{\bar t})_{\bar x},\ a z_{\bar x \bar t}] = (ap,\ z_{\bar x \bar t}^2] + (a p_{\bar t},\ z_{\bar t}^{(-1)} z_{\bar x \bar t}]. \tag{14}$$

We shall assume that the conditions

$$0 < c_1' \leqslant p \leqslant c_1, \qquad \left|\frac{\partial p}{\partial x}\right| \leqslant c_2, \qquad \left|\frac{\partial p}{\partial t}\right| \leqslant c_3, \tag{15}$$

are satisfied, where c_1', c_1, c_2, c_3 are positive constants. We shall then have

$$0 < c_1' \leqslant \frac{1}{a} \leqslant c_1, \qquad \left|\left(\frac{1}{a}\right)_{\bar x}\right| \leqslant c_2, \qquad \left|\left(\frac{1}{a}\right)_{\bar t}\right| \leqslant c_3. \tag{16}$$

Using the relation

$$\frac{1}{\check a}\,| a_{\bar t} | \leqslant a c_3 \leqslant c_3/c_1', \qquad p \leqslant 1 + Mh, \qquad M = M\,(c_2,\ c_1') > 0,$$

and Lemma 1 we obtain

$$\tau \,\| z_{\bar t} \|^2 + 0.5\, I \leqslant 0.5\,(1 + \tau c_3/c_1')\, \check I + (c_0 + M_1 \tau)\,\| z_{\bar t} \|^2 + \frac{\tau}{4c_0}\|\psi\|^2,$$
$$I = (a,\ z_{\bar x}^2], \qquad a\,(x,\ t_{-1/2}) = a\,(x, 0), \qquad M_1 = M\,(c_1',\ c_2,\ c_3), \tag{17}$$

where c_0 is an arbitrary positive constant. We have used the estimate

$$h\,(a p_{\bar t},\ z_{\bar t}^{(-1)} z_{\bar x \bar t}] \leqslant \frac{2c_3}{c_1'}\,\| z_{\bar t} \|^2.$$

If h is sufficiently small:

$$h \leqslant h_0, \quad \text{where } h_0 = h_0\,(c_0,\ c_1',\ c_2,\ c_3) > 0, \tag{18}$$

from (17) we have

$$I \leqslant (1 + c^*\tau)\, \check I + \frac{1}{2c_0}\tau\,\|\psi\|^2, \qquad c^* = c_3/c_1'. \tag{19}$$

From this we find (for $z(x, 0) = 0$)

$$I \leqslant M' (\overline{\|\psi\|})^2, \qquad \| z^{j+1} \|_0 \leqslant M \overline{\| \psi^{j+1} \|}, \quad M = M (c_1', c_2, c_3, c_0). \tag{20}$$

By analogy with Section 1 we can use the estimate

$$\tau (\psi, z_{\bar{t}}) = \tau (\psi, z)_{\bar{t}} - \tau (\psi_{\bar{t}}, \check{z}) \leqslant \tau (\psi, z)_{\bar{t}} + 0.5\tau c_0 \check{I} + \frac{l^2}{8c_0c'_1} \tau \| \psi_{\bar{t}} \|^2 \tag{21}$$

for $\tau (\psi, z_{\bar{t}})$. Then, instead of (19), we shall have

$$I \leqslant (1 + (c^* + c_0) \tau) \check{I} + 2\tau (\psi, z)_{\bar{t}} + \frac{l^2}{4c_0c'_1} \tau \| \psi_{\bar{t}} \|^2 \tag{22}$$

with the condition

$$h \leqslant h_0, \qquad h_0 = h_0 (c_1', c_2, c_3). \tag{23}$$

As usual from (22) we find

$$\| z^{j+1} \|_0 \leqslant M \, (\max_{1 \leqslant j' \leqslant j+1} \| \psi^{j'} \| + \overline{\| \psi_{\bar{t}}^{j+1} \|}) \quad \text{for} \quad z (x, 0) = 0, \tag{24}$$

$$\| z_{\bar{x}}^{j+1} \| \leqslant M \| z_{\bar{x}} (x, 0) \| \quad \text{for} \quad \psi = 0. \tag{25}$$

This proves the following theorem.

Theorem 10. If conditions (15) are satisfied, then for sufficiently small $h \leqslant h_0$ and any τ the scheme (10) is absolutely stable w.r.t. the initial data and w.r.t. the right-hand side ψ, so that estimates (20), (24), (25) hold.

Theorem 11. Let the conditions for which the scheme (6) has maximum order of approximation (12) be satisfied. Then the scheme (6) is uniformly convergent as h and $\| \tau \|_0 = \max_{\overline{\omega_\tau}} \tau_j$ tend independently to zero, so that for sufficiently small $h \leqslant h_0$ we have the estimate

$$\| y - u \|_0^{j+1} \leqslant M \, (h^4 + \| \tau \|_0^2), \tag{26}$$

where M is a positive constant which does not depend on the net.

4. Up to now we have been assuming that the functional $A\,[\mu\,(s)]$ satisfies conditions (4) only, and is otherwise completely arbitrary. It is not difficult to see that the pattern functional

$$A \, [\mu \, (s)] = \int_{-1}^{0} \mu \, (s) \, ds = \int_{-0.5}^{0.5} \mu(s - 0.5) ds \tag{27}$$

satisfies conditions (4). However, this functional is not always suitable for practical purposes. The most suitable functionals in calculations are the "discrete" functionals [13] which depend on the values of the function at a finite number of points. In particular, the functional

$$A[\mu(s)] = \tfrac{1}{6}[\mu(-1) + \mu(0)] + \tfrac{2}{3}\mu(-0.5) \tag{28}$$

satisfies conditions (4). In this case the coefficient $1/a$ is equal to

$$\frac{1}{a_i} = \frac{1}{6}(p_{i-1} + p_i) + \frac{2}{3}p_{i-1/2}, \qquad p_{i-1/2} = p(x_{i-1/2}, t), \tag{29}$$

where $x_{i-1/2} = x_i - 0.5\,h$. Theorems 10 and 11 are valid for the scheme (6) in which the coefficient a is calculated from formula (29).

We have restricted our attention to a scheme of order (4,2) for the simplest equation (1). Using (5), it is not difficult to show that the scheme of high order accuracy for the equation

$$c(x, t)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x, t)\frac{\partial u}{\partial x}\right) - q(x, t)\,u + f(x, t) \tag{30}$$

has the form

$$c(x, t_{j+1/2})\, y_{\bar{t}} = 0.5\,\Lambda(y + \check{y}) - \frac{h^2}{12}\Lambda(cp\, y_{\bar{t}} + pqy) - q^{j+1/2}y + \varphi, \tag{31}$$

where φ and Λy are given by formulae (7). With corresponding conditions Theorems 10 and 11 remain true for this scheme.

5. Similarly, we can construct a high accuracy scheme for the heat conduction equation with several space variables. Here we shall restrict our attention to the case of two dimensions ($p = 2$), see Section 2.

We shall consider the problem

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{2} L_\alpha u + f(x, t), \qquad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(\frac{1}{p_\alpha(x,t)}\frac{\partial u}{\partial x_\alpha}\right), \qquad 0 < x_\alpha < l_\alpha, \quad 0 < t \leqslant T, \tag{32}$$

$$u(x, t) = \mu(x, t) \text{ for } x_\alpha = 0, \quad x_\alpha = l_\alpha, \ \alpha = 1,2; \qquad u(x, 0) = u_0(x). \tag{33}$$

To simplify the argument we can without loss of generality take the net $\omega_h = \{(i_1h_1;\ i_2h_2) \in G\}$ to be square, i.e. $h_\alpha = l_\alpha/N_\alpha = h = \text{const.}$, $\alpha = 1, 2$. The net $\bar{\omega}_\tau$ is arbitrary. Let us define the net functions a_α with the help of the functional $A[\mu(s)]$ from Para. 1, putting

$$a_1 = a_1(x, \bar{t}) = 1/A\ [p_1(x_1 + sh, x_2, \bar{t})], \qquad \bar{t} = t - 0{,}5\tau,$$
$$a_2 = a_2(x, \bar{t}) = 1/A\ [p_2\ (x_1, x_2 + sh, \bar{t})],$$
$$\Lambda_\alpha y = (a_\alpha y_{\bar{x}_\alpha})_{x_\alpha}, \qquad \Lambda = \sum_{\alpha=1}^{p} \Lambda_\alpha.$$

By the argument of Paragraph 1 we can see that the initial scheme of order (4,2) of approximation has the form

$$y_{\bar{t}} = 0.5\,\Lambda\,(y + \check{y}) - \frac{h^2}{12}\sum_{\alpha=1}^{2}\Lambda_\alpha\,(p_\alpha y_{\bar{t}}) + \frac{h^2}{12}[\Lambda_1(p_1\Lambda_2\check{y}) + \Lambda_2(p_2\Lambda_1\check{y})] + \varphi, \tag{34}$$

$$\varphi = f^{j+1/2} + \frac{h^2}{12}\sum_{\alpha=1}^{2}(\Lambda_\alpha p_\alpha f)^{j+1/2}. \tag{35}$$

6. Let us find the generating scheme. Introducing the notation

$$\sigma_\alpha = \frac{1}{2}\left(1 - \frac{h^2}{6\tau}p_\alpha\right) \tag{36}$$

and replacing $E - \tau\sum_{\alpha=1}^{2}\Lambda_\alpha\sigma_\alpha$ by the product A_1A_2, where $A_\alpha = E - \tau\Lambda_\alpha\sigma_\alpha$, we obtain the generating scheme

$$y_{\bar{t}} = \sum_{\alpha=1}^{2}[\Lambda_\alpha\sigma_\alpha y + \Lambda_\alpha(1 - \sigma_\alpha)\check{y}] - \tau^2\Lambda_1\sigma_1\Lambda_2\sigma_2 y_{\bar{t}} + \tau\Lambda_1\left(\frac{1}{2} - \sigma_1\right)\Lambda_2\check{y} + \tau\Lambda_2\left(\frac{1}{2} - \sigma_2\right)\Lambda_1\check{y} + \varphi, \tag{37}$$

$$y = \mu \text{ for } x \in \gamma,\ t \in \bar{\omega}_\tau; \quad y(x, 0) = u_0(x) \text{ for } x \in \bar{\omega}_h. \tag{38}$$

The reduction of the generating scheme to one-dimensional alternating direction algorithms is achieved by analogy with the case of constant coefficients. Here we shall just give algorithm A (see Section 3):

$$\frac{y_{(1)} - \check{y}}{\tau} = \Lambda_1\sigma_1 y_{(1)} + \Phi, \qquad \frac{y_{(2)} - y_{(1)}}{\tau} = \Lambda_2\sigma_2 y_{(2)}, \qquad y^{j+1} = y_{(2)}, \tag{39}$$
$$\Phi = \sum_{\alpha=1}^{2}\Lambda_\alpha(1 - \sigma_\alpha)\check{y} + \tau[\Lambda_1(0.5 - \sigma_1)\Lambda_2\check{y} + \Lambda_2(0.5 - \sigma_2)\Lambda_1\check{y}] + \varphi.$$

7. Now let us discuss the basic problem, that of the stability and convergence of the generating scheme. Let u be a solution of the initial problem (32)-(33), y a solution of the difference problem (37)-(38). For the difference $z = y - u$ we have

$$z_{\bar{t}} - 0.5\,\Lambda z + \tau^2\Lambda_1\sigma_1\Lambda_2\sigma_2 z_{\bar{t}} = 0.5\,\Lambda\check{z} - \frac{h^2}{12}\sum_{\alpha=1}^{2}\Lambda_\alpha\,(p_\alpha z_{\bar{t}}) + \\ + \tau\Lambda_1\,(0.5-\sigma_1)\,\Lambda_2\check{z} + \tau\Lambda_2\,(0.5-\sigma_2)\,\Lambda_1\check{z} + \Psi, \qquad (x,\,t)\in Q_T, \tag{40}$$

$$z = 0 \text{ for } x\in\gamma, \quad t\in\bar{\omega}_\tau, \; z\,(x,\,0) = 0 \text{ for } x\in\bar{\omega}_h. \tag{41}$$

Multiplying (40) scalarly by $2\tau z_{\bar{t}}$, we obtain the basic energy identity

$$2\tau\,\|z_{\bar{t}}\|^2 + \sum_{\alpha=1}^{2}(a_\alpha,\,z^2_{\bar{x}_\alpha}]_\alpha + 2\tau^3\,(\Lambda_1\sigma_1\Lambda_2\sigma_2 z_{\bar{t}},\,z_{\bar{t}}) = \sum_{\alpha=1}^{2}(a_\alpha,\,\check{z}^2_{\bar{x}_\alpha}]_\alpha + \\ + \frac{h^2}{6}\tau\sum_{\alpha=1}^{2}((p_\alpha z_{\bar{t}})_{\bar{x}_\alpha},\,a_\alpha z_{\bar{x}_\alpha\bar{t}}]_\alpha + \frac{\tau h^2}{6}(\Lambda_1 p_1\Lambda_2\check{z} + \Lambda_2 p_2\Lambda_1\check{z},\,z_{\bar{t}}) + 2\tau\,(\Psi,\,z_{\bar{t}}). \tag{42}$$

We shall assume that

$$0 < c_1' \leqslant p_\alpha \leqslant c_1, \quad \left|\frac{\partial p_\alpha}{\partial x_\beta}\right| \leqslant c_2, \quad \left|\frac{\partial p_\alpha}{\partial t}\right| \leqslant c_3, \quad \left|\frac{\partial^2 p_\alpha}{\partial x_\alpha\partial x_\beta}\right| \leqslant c_4, \quad \alpha,\,\beta = 1,2. \tag{43}$$

Constants which depend only on $c_1',\,c_1,\,c_2,\,c_3,\,c_4$, are denoted by M.

8. As in the one-dimensional case we find

$$A_1 = \frac{h^2\tau}{6}((p_\alpha z_{\bar{t}})_{\bar{x}_\alpha},\,a_\alpha z_{\bar{x}_\alpha\bar{t}}]_\alpha \leqslant \frac{2}{3}\tau\,(1+M_1 h)\,\|z_{\bar{t}}\|^2. \tag{44}$$

Now let us estimate the other terms in (42).

Consider the expression

$$A_2 = \frac{\tau h^2}{6}(\Lambda_1 p_1\Lambda_2\check{z},\,z_{\bar{t}}) = -\frac{\tau h^2}{6}((p_1\Lambda_2\check{z})_{\bar{x}_1},\,a_1 z_{\bar{x}_1\bar{t}}). \tag{45}$$

Lemma 5. If conditions (43) are satisfied, then

$$A_2 \leqslant \frac{\tau h^2}{12}\Big[(a_1 a_2 p_1^{(-1_1)},\,z^2_{\bar{x}_1\bar{x}_2})_{\bar{t}} - \tau\,(a_1 a_2 p_1^{(-1_1)},\,z^2_{\bar{x}_1\bar{x}_2\bar{t}})\Big] + M_2\tau h^2\,\|\check{z}_{\bar{x}_2}\|\,\|z_{\bar{x}_1\bar{x}_2\bar{t}}\|, \\ M_2\tau h^2\,\|\check{z}_{\bar{x}_2}\|\,\|z_{\bar{x}_1\bar{x}_2\bar{t}}\| \leqslant 4M_2\tau\,\|\check{z}_{\bar{x}_2}\|\,\|z_{\bar{t}}\| \leqslant c_0\tau\,\|z_{\bar{t}}\|^2 + \frac{4M_2^2}{c_0}\tau\,(\check{a}_2,\,\check{z}^2_{\bar{x}_2}]_2. \tag{46}$$

Estimate (46) follows from the expression

$$A_2 = \frac{\tau h^2}{6}\Big\{\big(a_2\check{z}_{\bar{x}_1\bar{x}_2} + a_{2\bar{x}_1}\check{z}^{(-1_1)}_{\bar{x}_2},\,a_1 p_1^{(-1_1)} z_{\bar{x}_1\bar{x}_2\bar{t}} + (a_1 p_1^{(-1_1)})_{\bar{x}_2}\,z^{(-1_2)}_{\bar{x}_1\bar{t}}\big) + \\ + \big(a_2\check{z}_{\bar{x}_2},\,p_{1\bar{x}_1}a_1 z_{\bar{x}_1\bar{x}_2\bar{t}} + (a_1 p_{1\bar{x}_1})_{\bar{x}_2}\,z^{(-1_1)}_{\bar{x}_1\bar{t}}\big)\Big\},$$

which is obtained from (45) after the use of Green's formula for the variable x_2.

9. *Lemma 6.* For any σ_α, $\alpha = 1, 2$ we have the estimate

$$\begin{aligned} A_3 = 2\tau^3 (\Lambda_1\sigma_1\Lambda_2\sigma_2 z_{\bar{t}}, z_{\bar{t}}) \geqslant 2\tau^3 (a_1 a_2 \sigma_1^{(-1_1)}\sigma_2, z^2_{\bar{x}_1\bar{x}_2\bar{t}}) - \\ - c_0\tau^3 (a_1 a_2, z^2_{\bar{x}_1\bar{x}_2\bar{t}}) - M_3\tau h \| z_{\bar{t}} \|^2 - M_4\tau (I + \check{I}), \end{aligned} \tag{47}$$

where

$$I = \sum_{\alpha=1}^{2} (a_\alpha, z^2_{\bar{x}_\alpha}]_\alpha.$$

Using Green's formula for x_1 and x_2 after a number of transformations we find

$$\begin{aligned} A_3 = 2\tau^3\{(a_2 (\sigma_2 v)_{\bar{x}_2}, (a_1\sigma_{1\bar{x}_1})_{\bar{x}_2} v_{\bar{x}_1}) + (a_2 (\sigma_2 v)_{\bar{x}_1}, (a_1\sigma_{1\bar{x}_1})^{(-1_2)} v_{\bar{x}_1\bar{x}_2}) + \\ + ((a_2\sigma_2 v_{\bar{x}_2} + a_2\sigma_{2\bar{x}_2} v^{(-1_2)})_{\bar{x}_1}, a_1\sigma_1^{(-1_1)} v_{\bar{x}_1\bar{x}_2} + (a_1\sigma_1^{(-1_1)})_{\bar{x}_2} v_{\bar{x}_1}^{(-1_2)}) \geqslant \\ \geqslant 2\tau^3 (a_1 a_2 \sigma_1^{(-1_1)}\sigma_2, v^2_{\bar{x}_1\bar{x}_2}) - \underline{M_5\tau^3 \| \sigma_1 \|_0 \| \sigma_2 \|_0 \| v_{\bar{x}_1} \| \| v_{\bar{x}_2} \|} - \\ - M_6 h^2\tau^2 (\| \sigma_1 \|_0 + \| \sigma_2 \|_0) \| v_{\bar{x}_1\bar{x}_2} \| (\| v_{\bar{x}_1} \| + \| v_{\bar{x}_2} \|) - M_7\tau h^4 \| v \| \| v_{\bar{x}_1\bar{x}_2} \|. \end{aligned} \tag{48}$$

The underlined expression is maximised as follows:

$$\text{a)} \quad \leqslant M_4\tau (I + \check{I}) \text{ for } \sigma_\alpha > 0, \quad \alpha = 1,2; \tag{49}$$

$$\text{b)} \quad \leqslant M_3' \tau h \| z_{\bar{t}} \|^2 \text{ for } \sigma_\alpha < 0. \tag{50}$$

To estimate the last two terms in (48) we must examine separately: (a) $0 \leqslant \sigma_\alpha \leqslant 0.5$, б) $\sigma_\alpha < 0$ and, therefore, $\| \sigma_\alpha \|_0 \leqslant h^2 c_1/12\tau$.

We also need the estimates

$$\begin{aligned} 2\sigma_1^{(-1_1)}\sigma_2 + 1 - \sigma_1^{(-1_1)} - \sigma_2^{(-1_2)} = 0.5 [1 + (1 - 2\sigma_1^{(-1_1)}) (1 - 2\sigma_2)] + (\sigma_2 - \sigma_2^{(-1_2)}) \geqslant \\ \geqslant 0.5 - M_7 h^3/\tau, \end{aligned} \tag{51}$$

$$M h^3\tau^2 \| z_{\bar{x}_1\bar{x}_2\bar{t}} \|^2 \leqslant \dot{M}\tau^2 h \| z_{\bar{x}_1\bar{x}_2\bar{t}} \| \| z_{\bar{t}} \| \leqslant c_0\tau^3 \| z_{\bar{x}_1\bar{x}_2\bar{t}} \|^2 + M_8\tau h^2 \| z_{\bar{t}} \|^2. \tag{52}$$

10. Now, collecting together all the estimates (44), (46), (47), (51), (52) and choosing c_0 we obtain from (42) for sufficiently small h and τ,

$$h \leqslant h_0, \quad \tau \leqslant \tau_0, \tag{53}$$

the energy identity:

$$\begin{aligned} (1 - M_9\tau_{j+1}) I^{j+1} \leqslant M_{10} \sum_{j'=2}^{j+1} \tau_{j'} I^{j'-1} + \frac{h^2}{12} (a_1 a_2 (p_1^{(-1_1)} + p_2^{(-1_2)}), z^2_{\bar{x}_1\bar{x}_2})^{j+1} + \\ + (1 + M_{10}\tau_1) I(0) + M_{11} (\overline{\| \Psi^{j+1} \|})^2. \end{aligned} \tag{54}$$

For sufficiently small $h \leqslant h_0$ we have $a_\alpha p_\alpha^{(-1_\alpha)} \leqslant 1 + M_{12}h$. We use the inequality

$$h^2 \| z_{\bar{x}_1 \bar{x}_2} \|^2 \leqslant \frac{4}{c_1'} I.$$

We arrive at the following inequality:

$$\| z^{j+1} \| \leqslant M_{12} \| z_{\bar{x}} (x, 0) \| + M_{13} \| \overline{\Psi^{j+1}} \| \text{ for } h \leqslant h_0, \quad \tau \leqslant \tau_0,$$

where M_{12}, M_{13} are positive constants which depend only on c_1', c_1, ..., c_4, l_1, l_2.

11. We have thus proved the following theorems.

Theorem 12. If conditions (45) are satisfied, then the generating scheme (34) is unconditionally stable w.r.t. the right-hand side Ψ and the initial data, so that for sufficiently small h and τ we have the estimate

$$\| z^{j+1} \| + \| z_{\bar{x}} \|^{j+1} \leqslant M \| z_{\bar{x}}(x, 0) \| + M' \| \overline{\Psi^{j+1}} \|, \quad h \leqslant h_0, \ \| \tau \|_0 \leqslant \tau_0, \tag{55}$$

where h_0, τ_0, M and M' are positive constants which depend only on c_1', c_1, c_2, c_3, c_4, l_1, l_2.

Theorem 13. If the conditions for which

$$\| \Psi \| = O \left(|h|^4 + \tau^2 \right), \tag{56}$$

and conditions (43) are satisfied, then scheme (34) converges as h and τ tend independently to zero so that, for sufficiently small h and τ we have the estimate

$$\| y^{j+1} - u^{j+1} \| \leqslant M \left(|h|^4 + \| \tau^2 \|_{j+1} \right) \text{ for } h \leqslant h_0, \quad \tau \leqslant \tau_0$$

where M are positive constants which do not depend on the choice of the net. For a hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = \sum_{\alpha=1}^{p} L_\alpha u, \quad L_\alpha u = \frac{\partial^2 u}{\partial x^2_\alpha}$$

also we can write down economical high accuracy schemes.

Using the above methods we can show that the generating schemes

$$Ay = 2\check{y} - (E + 0.5\,\tau^2) \Lambda \check{\check{y}}, \qquad A = \prod^{p} A_\alpha, \qquad A_\alpha = E - \sigma_\alpha \tau^2 \Lambda_\alpha,$$

$$Ay_{\bar t} = (E - 0.5\,\tau^2\Lambda)\overset{\alpha=1}{\vphantom{x}}\check{y}_{\bar t} + \tau\Lambda\check{y}$$

are absolutely stable and converge in the mean at a rate $O(\tau^2 + |h|^4)$. We shall examine the question of economical schemes for multidimensional hyperbolic equations separately.

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