HOMOGENEOUS DIFFERENCE SCHEMES ON NON-UNIFORM NETS FOR EQUATIONS OF PARABOLIC TYPE*

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The basic problems of the theory of homogeneous difference schemes for linear, quasi-linear and non-linear equations of parabolic type have been studied in a number of works ([1]-[5]). The stability, convergence, and also estimates of the rate of convergence (order of accuracy) of several families of homogeneous difference schemes in the classes of continuous and discontinuous coefficients of the differential equation have been established. In [6] attention was paid to the fact that on an arbitrary sequence of non-uniform nets difference schemes which have second order approximation on uniform nets have only first order approximation. For this reason the problem of the order of accuracy on non-uniform nets requires special study. A family of homogeneous difference schemes on non-uniform nets for the equation

$$L^{(k,q,f)}u = \frac{d}{dx}\left(k(x)\frac{du}{dx}\right) - q(x)u + f(x) = 0 \tag{1.B}$$

was given in [6]. The definition of this family was the same as in the case of uniform nets with homogeneous difference schemes, studied in [7]. It was shown that the order of accuracy of these schemes on non-uniform nets is equal to the order of their accuracy on uniform nets both in the case of continuous coefficients in the differential equation and in the case of discontinuous coefficients. It is interesting to note that this result is obtained if we use *a priori* estimates of the same type as in the case of discontinuous coefficients [7] and a uniform net. The effective characteristic of the net is the mean square step

$$\|h\|_2 = \left(\sum_{i=1}^{N} h_i^2 h_i\right)^{1/2}, \tag{2.B}$$

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so that the estimate $\|z\|_0 = O(h^2)$ on a uniform net corresponds to the estimate $\|z\|_0 = O(\|h\|_2^2)$ on a non-uniform net.

In this article we study homogeneous difference schemes for parabolic type equations with one space variable. In Section 1 we consider a new family of homogeneous differences for the equation (1.B). On special sequences of nets $\omega_h(k)$ which depend on the choice of the coefficients k, q, f this family has second order accuracy $(\|y - u\|_0 = O(\|h\|_2^2))$ in the class of continuous coefficients.

In Section 2 we consider homogeneous difference schemes for the linear and quasi-linear equations

$$c(x, t)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x, t)\frac{\partial u}{\partial x}\right) + r(x, t)\frac{\partial u}{\partial x} - q(x, t)u + f(x, t), \quad (2.\mathrm{B})$$

$$c(x, t)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x, t, u)\frac{\partial u}{\partial x}\right) + f\left(x, t, u, \frac{\partial u}{\partial x}\right) \quad (3.\mathrm{B})$$

and calculate the approximation error of these schemes on non-uniform nets.

Section 3 is devoted to *a priori* estimates. Among the estimates obtained in this section Theorem 5 should be noted, for even in the case of uniform nets it essentially strengthens the results obtained in [1] and [3]. Using the special representation of the approximation error of the schemes found in Section 2 and the *a priori* estimates of Section 3 we prove, in Section 4, a number of theorems about the order of accuracy of homogeneous difference schemes for (2.B) and (3.B) on a sequence of non-uniform nets. The results of Section 4 lead to the conclusion that the order of accuracy of the given schemes remains unchanged on transferring to non-uniform nets (by analogy with [6]).

In Point 5 of Section 4 we examine a homogeneous difference scheme for a system of parabolic equations which is "economical" with respect to the number of operations.

Each section has its own numbering for formulae; when referring to the formulae of another section we use a dual system. For example (2.3) is formula (3) of Section 2.

1. A stationary equation

1. *Homogeneous difference schemes on a non-uniform net.* In [6] we studied homogeneous difference schemes on a non-uniform net for the

stationary heat-conduction equation

$$L^{(k,q,f)}u = \frac{d}{dx}\left(k(x)\frac{du}{dx}\right) - q(x)\,u + f(x) = 0, \quad 0<x<1,$$
$$u(0) = u_1, \qquad u(1) = u_2,$$
$$k(x) \geqslant c_1 > 0, \qquad q(x) \geqslant 0 \qquad (c_1 = \text{const.}). \tag{1}$$

Let $\bar{\omega}_h = \{x_i \in [0, 1], 0 \leqslant i \leqslant N\}$ be an arbitrary difference net on the segment $0 \leqslant x \leqslant 1$, where $x_0 = 0$, $x_N = 1$. The step of the net $h_i = x_i - x_{i-1} > 0$ $(0 < i \leqslant N)$ satisfies only the natural normalisation condition

$$\sum_{i=1}^{N} h_i = 1, \tag{2}$$

and is otherwise an arbitrary net function. We shall use a notation without suffixes (cf. [6]):

$$h = h_i, \quad h_+ = h_{i+1}, \ \hbar = 0.5(h + h_+), \quad y = y(x) = y_i,$$
$$y^{(+1)} = y(x + h_+) = y_{i+1}, \quad y^{(-1)} = y(x - h) = y_{i-1}, \quad y_{\bar{x}} = (y - y^{(-1)})/h,$$
$$y_x = (y^{(+1)} - y)/h_+, \quad y_{\hat{x}} = (y^{(+1)} - y)/\hbar = \frac{h_+}{\hbar}\, y_x, \quad y_{\check{x}} = \frac{h}{\hbar}\, y_{\bar{x}}.$$

In [6] we considered canonical schemes of the form

$$L_h^{(k,q,f)}y = (a y_{\bar{x}})_{\hat{x}} - dy + \varphi = 0, \qquad y_0 = u_1, \quad y_N = u_2. \tag{3}$$

The coefficients a, d, φ were calculated with the help of the same pattern functionals $A\,[\bar{k}(s)]\,(-1 \leqslant s \leqslant 0)$, $D\,[\bar{q}(s)]\ (-0.5 \leqslant s \leqslant 0.5)$, $F\,[\bar{f}(s)]\ (-0.5 \leqslant s \leqslant 0.5)$, as in the case of uniform nets (cf. [7]), from the formulae

$$a = A\,[k(x + sh)], \tag{4}$$

$$d = D\,[q(x + (s + \Delta)\,\hbar)], \qquad \varphi = F\,[f(x + (s + \Delta)\,\hbar)],$$
$$\Delta = (h_+ - h)/4\hbar. \tag{5}$$

The family of homogeneous schemes (3) is defined when the class of the pattern functionals A, D, F is given by the law (4)-(5) for calculating the coefficients of the scheme, a, d, φ on an arbitrary non-uniform net.

In this article we shall consider another family of homogeneous schemes (3) which differs from the family of [6] in that it has another law for calculating a, d, φ on a non-uniform net. We shall assume that A, D, F satisfy the following conditions (cf. [6]):

1) $A\ [\bar{k}(s)]$ is a non-decreasing homogeneous functional of the first degree having a second differential,

2) $D\ [\bar{q}(s)]$ and $F\ [\bar{f}(s)]$ are linear non-negative functionals ($F\ [f] \geqslant 0$ when $f \geqslant 0$),

3) $A\ [1] = 1,\ A_1\ [s] = -0.5,\ D\ [1] = F\ [1] = 1,\ D\ [s] = F\ [s] = 0$ (6)

(necessary conditions of the second order approximation on a uniform net).

Moreover, we assume that D and F satisfy the additional requirements:

4)
$$\begin{aligned} &F\ [\eta_0^-(s)] = \tfrac{1}{2}, \qquad F\ [\pi_0(s)] = 0, \\ &D\ [\eta_0^-(s)] = \tfrac{1}{2}, \qquad D\ [\pi_0(s)] = 0. \end{aligned} \tag{7}$$

We have used here the following notation (cf. [7], Section 1, Point 11):

$$\eta_\theta^-(s) = \begin{cases} 1 & \text{for } s < \theta, \\ 0 & \text{for } s \geqslant \theta; \end{cases} \quad \pi_\theta(s) = \begin{cases} 1, & s = \theta, \\ 0, & s \neq \theta, \end{cases}$$

where θ is an arbitrary number. Introducing the function

$$\eta_\theta^+(s) = \begin{cases} 0 & \text{for } s \leqslant \theta, \\ 1 & \text{for } s > \theta, \end{cases}$$

we shall have

$$\eta_\theta^-(s) + \eta_\theta^+(s) + \pi_\theta(s) = 1. \tag{8}$$

The condition $F\ [\pi_0(s)] = 0$ means that $F\ [\bar{f}(s)]$ does not depend on $\bar{f}(0)$ (the functional F is regular at the point $s = 0$, cf. [7], Section 1, Point 11).

We shall calculate the coefficient φ from the formula

$$\varphi = F\ [f^*(s)],$$

$$f^*(s) = \frac{h}{\hbar} f(x + sh)\, \eta_0^-(s) + \frac{h_+}{\hbar} f(x + sh_+)\, \eta_0^+(s) + f(x)\, \pi_0(s),\ -0.5 \leqslant s \leqslant 0.5.$$

Due to the linearity of F and the condition $F\ [\pi_0(s)] = 0$

$$\varphi = \varphi(x) = \frac{h}{\hbar} F\ [f(x + sh)\, \eta_0^-(s)] + \frac{h_+}{\hbar} F\ [(f(x + sh_+)\, \eta_0^+(s)]. \tag{9}$$

The coefficient d is defined similarly:

$$d = d(x) = \frac{h}{\hbar} D\,[q(x + sh)\,\eta_0^-(s)] + \frac{h_+}{\hbar} D\,[q(x + sh_+)\,\eta_0^+(s)]. \tag{10}$$

From (6) and (7) we have

$$F\,[\eta_0^+(s)] = \tfrac{1}{2}, \qquad F\,[s\eta_0^+(s)] = -F\,[s\eta_0^-(s)]. \tag{11}$$

For, using (8), we find

$$F\,[\eta_0^+(s)] = F\,[1 - \eta_0^-(s) - \pi_0(s)] = F\,[1] - F\,[\eta_0^-(s)] = \tfrac{1}{2},$$
$$F\,[s\eta_0^+(s)] = F\,[s] - F\,[s\eta_0^-(s)] - F\,[s\pi_0(s)] = -F\,[s\eta_0^-(s)],$$

since $F\,[s] = 0$, $F\,[s\pi_0(s)] = F\,[0] = 0$.

If the net $\bar{\omega}_h$ is uniform, i.e. $h = h_+ = \hbar$, then formulae (5) and (9) have the same form:

$$\varphi = F\,[f(x + sh)].$$

In fact, in this case it follows from (9) that

$$\varphi = F\,[f(x + sh)\,(\eta_0^-(s) + \eta_0^+(s))] = F\,[f(x + sh)\,(1 - \pi_0(s))] = \\ = F\,[f(x + sh)],$$

since $F\,[f(x + sh)\,\pi_0(s)] = F\,[f(x)\,\pi_0(s)] = f(x)\,F\,[\pi_0(s)] = 0$. If $F\,[\pi_0(s)] \neq 0$, then formula (8) gives

$$\varphi = F\,[f(x + sh)] - f(x)\,F\,[\pi_0(s)].$$

For the special case

$$F = F^*\,[\mu] = \int_{-0.5}^{0.5} \mu(s)\,ds$$

schemes (5) and (9) are the same on an arbitrary net $\bar{\omega}_h$:

$$\varphi^* = F^*\,[f(x + (s + \Delta)\,\hbar)] = \frac{1}{\hbar} \int_{x-0.5h}^{x+0.5h_+} f(\xi)\,d\xi =$$
$$= \frac{h}{\hbar} \int_{-0.5}^{0} f(x + sh)\,ds + \frac{h_+}{\hbar} \int_{0}^{0.5} f(x + sh_+)\,ds,$$

i.e.

$$\varphi^* = \frac{h}{\hbar} F^* \left[f(x + sh)\, \eta_0^-(s) \right] + \frac{h_+}{\hbar} F^* \left[f(x + sh_+)\, \eta_0^+(s) \right].$$

But in the general case formulae (5) and (9) give different results. It is sufficient to give the example

$$F\,[\bar{f}(s)] = \frac{1}{2}\left[\bar{f}\left(-\frac{1}{2}\right) + \bar{f}\left(\frac{1}{2}\right)\right].$$

Formula (5) gives at once

$$\varphi = \frac{1}{2}\left[f(x - 0.5h) + f(x + 0.5h_+)\right]. \tag{12}$$

Using formula (9) we find

$$\varphi = \frac{h}{2\hbar} f(x - 0.5h) + \frac{h_+}{2\hbar} f(x + 0.5h_+). \tag{13}$$

Expressions (12) and (13) are equal only on a uniform net (f is an arbitrary function).

Thus, the initial family of schemes whose coefficients are defined by formulae (4), (9) and (10) is different from the family of schemes considered in [6].

2. *Approximation error*. Let us find an expression for the approximation error of the scheme

$$L_h^{(k,q,f)}\, y = (a y_{\bar{x}})_{\hat{x}} - dy + \varphi = 0, \qquad y_0 = u_1, \qquad y_N = u_2, \tag{14}$$

where a, d, φ are coefficients defined by formulae (4), (9) and (10).

Let $u(x)$ be an arbitrary function satisfying the requirements $u \in C^{(2)}$, $(ku')' \in C^{(1,1)}$ and $k, q, f \in C^{(1,1)}$. Then, by analogy with [6], we find

$$\begin{gathered} (a u_{\bar{x}})_{\hat{x}} = (ku')' + (\mu_a)_{\hat{x}} + \psi_a^*, \\ \mu_a = a u_{\bar{x}} - \overline{ku'} + \frac{1}{8} h^2\, \overline{(ku')''}, \qquad \psi_a^* = O(h^2) + O(h_+^2), \end{gathered} \tag{15}$$

where $\bar{k} = k(x - 0.5\,h)$ etc. Let us expand φ in powers of h and h_+:

$$\begin{aligned} \varphi = \frac{h}{\hbar} F\,[(f(x) + shf'(x))\, \eta_0^-(s)] &+ \\ + \frac{h_+}{\hbar} F\,[(f(x) + sh_+ f'(x))\, \eta_0^+(s)] &+ O(h^2) + O(h_+^2). \end{aligned} \tag{16}$$

From this, and from (6), (11), it follows that

$$\varphi = f(x) - (h_+^2 - h^2) f'(x) F[s\eta_0^-(s)]/\hbar + O(h^2) + O(h_+^2),$$

or

$$\varphi - f = -(h^2\bar{f}')_{\hat{x}} F[s\eta_0^-(s)] + O(h^2) + O(h_+^2). \tag{17}$$

In [6] we obtained for the scheme (5) the expression

$$\varphi - f = \tfrac{1}{8}(h^2\bar{f}')_{\hat{x}} + O(h^2) + O(h_+^2).$$

We note that $F[s\eta_0^-(s)] = -\frac{1}{8}$, for example in the case

$$F = F^*[\mu] = \int_{-0.5}^{0.5} \mu(s)\,ds \quad \text{or} \quad F[\mu] = \tfrac{1}{2}\left[\mu\left(-\tfrac{1}{4}\right) + \mu\left(\tfrac{1}{4}\right)\right].$$

By analogy with (17) we find

$$(d - q)u = -(h^2\overline{q'u})_{\hat{x}} D[s\eta_0^-(s)] + O(h^2) + O(h_+^2). \tag{18}$$

Using (15), (17), (18) we obtain for the approximation error of the scheme (14)

$$\psi = L_h^{(k,q,f)} u - L^{(k,q,f)} u \tag{19}$$

the expression

$$\begin{gathered} \psi = \mu_{\hat{x}} + \psi^*, \qquad \psi^* = O(h^2) + O(h_+^2), \\ \mu = au_{\bar{x}} - \overline{ku'} + \tfrac{1}{8} h^2 \overline{(ku')''} + h^2\overline{q'u} D[s\eta_0^-(s)] - \overline{h^2 f'} F[s\eta_0^-(s)]. \end{gathered} \tag{20}$$

If A has a second differential, then, from conditions (6)

$$a = \bar{k} + O(h^2) \quad \text{and} \quad \mu = O(h^2).$$

If $u = u(x)$ is the solution of the differential equation (1) then $(ku')'' = qu)' - f'$ and

$$\mu = au_{\bar{x}} - \overline{ku'} + h^2\left\{\left(-\tfrac{1}{8} + D[s\eta_0^-(s)]\right)\overline{q'u} - \left(\tfrac{1}{8} + F[s\eta_0^-(s)]\right)\overline{f'} + \tfrac{1}{8}\overline{qu'}\right\} \tag{21}$$

3. *The case of discontinuous coefficients.* Let us now assume that k, q and f have a discontinuity of the first kind at the node $x \in \omega_h$. For simplicity we shall assume for the time being that there is only one point of discontinuity. Put $f_1 = f(x - 0)$, $f_r = f(x + 0)$. Let $u(x)$

satisfy the coupling conditions $[u] = u_r - u_l = 0$, $[ku'] = 0$, at the point $x \in \omega_h$ and suppose that to the left and to the right of this point the conditions of Point 2 are satisfied.

We show that at the point of discontinuity $x \in \omega_h$ the representation (20) is valid if the approximation error ψ is defined as

$$\psi = L_h^{(k,q,f)}u - \widetilde{L^{(k,q,f)}}u, \tag{22}$$

where

$$\widetilde{L^{(k,q,f)}}u = \frac{h}{2\hbar}(L^{(k,q,f)}u)_l + \frac{h_+}{2\hbar}(L^{(k,q,f)}u)_r.$$

If $u = u(x)$ is the solution of equation (1), then $\widetilde{L^{(k,q,f)}}u = 0$. When $(L^{(k,q,f)}u)_l = (L^{(k,q,f)}u)_r$ the expression (19) for ψ follows from (22).

It is easy to see that the definition (22) is suitable by finding the expansion of $(au_{\bar{x}})_{\hat{x}}$, d, φ in powers of h and h_+ in the neighbourhood of the point of discontinuity of the coefficients of the equation, $x \in \omega_h$. For

$$\overline{(ku')} = (ku')_l - 0.5h\,(ku')'_l + \frac{1}{8}h^2\,(ku')''_l + O(h^3),$$
$$\overline{(ku')}^{(+1)} = (ku')_r + 0.5h_+\,(ku')'_r + \frac{1}{8}h_+^2\,(ku')''_r + O(h_+^3).$$

It follows from this and from the condition $[ku'] = (ku')_r - (ku')_l = 0$ that

$$\widetilde{(ku')'} = \overline{(ku')}_{\hat{x}} - \left(\frac{1}{8}h^2\,\overline{(ku')''}\right)_{\hat{x}} + O(h^2) + O(h_+^2),$$
$$(au_{\bar{x}})_{\hat{x}} = \widetilde{(ku')'} + \left(au_{\bar{x}} - \overline{ku'} + \frac{1}{8}h^2\,\overline{(ku')''}\right)_{\hat{x}} + O(h^2) + O(h_+^2).$$

Then expanding f in powers of h and h_+ in the neighbourhood of the point $x \in \omega_h$: $f(x + sh) = f_l + shf'_l + O(h^2)$, $s < 0$; $f(x + sh_+) = f_r + sh_+f'_r + O(h_+^2)$, $s > 0$, we find

$$\varphi = \frac{h}{\hbar}\{f_l F[\eta_0^-] + hf'_l F[s\eta_0^-(s)]\} + \frac{h_+}{\hbar}\{f_r F[\eta_0^+] + h_+f'_r F[s\eta_0^+(s)]\} + O(h^2) + O(h_+^2).$$

After substituting (7) and (11) in here we obtain

$$\varphi = \tilde{f} - (h^2\bar{f}')_{\hat{x}}\,F\,[s\eta_0^-(s)] + O(h^2) + O(h_+^2), \tag{23}$$

where

$$\widetilde{f} = \frac{h}{2\hbar} f_l + \frac{h_+}{2\hbar} f_r.$$

Similarly we find

$$(d - \widetilde{q})\, u = -\,(h^2 \overline{q'u})_{\hat{x}}\, D\, [s\eta_0^-(s)] + O\,(h^2) + O\,(h_+^2). \tag{24}$$

We can see as a result that the formula

$$\psi = L_n^{(k,\,q,\,f)} u - \widetilde{L^{(k,\,q,\,f)} u} = \mu_{\hat{x}} + \psi^*, \tag{25}$$

is valid, where μ is defined according to formula (20) and $\psi^* = O(h^2) + O(h_+^2)$.

When studying difference schemes for parabolic equations we shall make use of another representation:

$$\psi = \mu^*_{\hat{x}} + \psi^*, \quad \psi^* = O\,(h^2) + O\,(h_+^2), \quad \mu^* = h^2 \overline{\mu}_0^*, \quad \overline{\mu}_0^* = \mu_0\,(x - 0.5h),$$
$$\mu_0^* = \beta u' + \tfrac{1}{8}(ku')'' + q'uD\,[s\eta_0^-(s)] - f'F\,[s\eta_0^-(s)], \tag{26}$$

where

$$\beta = \frac{k''}{2}\left(A_1\,[s^2] - \tfrac{1}{4}\right) + \frac{(k')^2}{k} A_2\,[s].$$

These formulae have been obtained on the assumption that $k \in Q^{(2,1)}$ and the functional $A[\bar{k}(s)]$ has a third differential. Since it is elementary we shall not spend time on the derivation of these formulae.

In the case of the schemes of [6], generally speaking, the representation (25) does not apply if $x \in \omega_h$ is a point of discontinuity of the coefficients k, q, f of the differential equation. In this connection Note 2 on p. 829 of the article [6] needs to be made more precise.

4. *On the order of accuracy on non-uniform nets*. Since all the *a priori* estimates obtained in [1] are still in force and the expressions for ψ in [6] are of the same type as here, the theorems of [6] concerning the order of accuracy on non-uniform nets are valid also for the family of schemes (14), defined by the formulae (4), (9), (10) and conditions (7), Suppose that we are given coefficients k, q, f with a finite number of discontinuities at the points $x = \xi_\nu$, $\nu = 1, 2, \ldots, \nu_0$. We can always construct a non-uniform net ω_h such that the points $\xi_\nu \in (0, 1)$ are its nodes, i.e. $\xi_\nu = x_{n_\nu}$ $(\theta_\nu = 0)$, $0 < n_\nu < N$. We denote the sequence of nets $\bar{\omega}_h$ depending on the choice of the coefficients k, q, f and

constructed in this way by $\bar{\omega}_h(k)$. Using the methods of [6] and the representation (25) on the sequence of nets $\bar{\omega}_h(k)$ it is not difficult derive the following theorem.

Theorem 1. If $k,\ q,\ f \in Q^{(1,1)}$ then any scheme (14) of the initial family n. 1 has second order accuracy on the sequence of nets $\bar{\omega}_h(k)$ such that

$$\| y - u \|_0 \leqslant M \| h \|_2^2,$$

where y is the solution of problem (14), u is the solution of problem (1), M is a positive constant which is independent of the net and $\| h \|_2$ is the mean square step

$$\| h \|_2 = \left(\sum_{i=1}^{N} h_i^2 \hbar_i \right)^{1/2}.$$

To prove the theorem we need the *a priori* estimate

$$\| z \|_0 = \| y - u \|_0 \leqslant M \{ \| \mu^* \|_1 + \| \psi^* \|_3 \}$$

obtained in [6] and the representation (25) which is valid at all nodes of the net $\omega_h(k)$.

It must be emphasised that the only characteristic of the net is the mean square step $\| h \|_2$.

2. Parabolic equations

Let us now study homogeneous difference schemes for parabolic equations.

1. *Linear heat conduction equation.* We begin with the homogeneous difference schemes for the linear heat-conduction equation. Let us pose the initial problem. Suppose that the region $\bar{\Pi} = (0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant T)$ is given. It is required to find in $\bar{\Pi}$ the solution of the problem

$$c\,(x,\,t)\,\frac{\partial u}{\partial t} = L^{(k,\,q,\,f)} u = \frac{\partial}{\partial x}\left(k\,(x,\,t)\,\frac{\partial u}{\partial x}\right) - q\,(x,\,t)\,u + f\,(x,\,t), \tag{1}$$

$$u\,(0,\,t) = u_1\,(t), \qquad u\,(1,\,t) = u_2\,(t), \tag{2}$$

$$u\,(x,\,0) = u_0\,(x), \tag{3}$$

$$0 < c_1 \leqslant k\,(x,\,t), \qquad 0 < c_2 \leqslant c\,(x,\,t), \qquad 0 \leqslant q\,(x,\,t), \tag{4}$$

where c_1, c_2 are positive constants. Let $\Gamma_\nu\{x=\xi_\nu(t)\ (1\leqslant\nu\leqslant\nu_0)\}$ be a finite number of differentiable, non-intersecting curves. The coefficients k, q, f can have discontinuities of the first kind only on the curves $\Gamma_\nu(x = \xi_\nu(t),\ \nu = 1, 2, \ldots, \nu_0)$. If $k(x, t)$ has a discontinuity for $x = \xi_\nu(t)$ then the coupling conditions

$$[u]_\nu = u(\xi_\nu(t)+0, t) - u(\xi_\nu(t)-0, t) = 0, \quad \left[k\frac{\partial u}{\partial x}\right]_\nu = 0, \qquad x = \xi_\nu(t) \tag{5}$$

are satisfied.

As usual (cf. [5]) we put

$$\Delta_\nu = (\xi_\nu(t) < x < \xi_{\nu+1}(t),\ 0 < t \leqslant T), \qquad \nu = 0, 1, 2, \ldots, \nu_0,$$

$$\xi_0(t) = 0, \qquad \xi_{\nu_0+1}(t) = 1,$$

$$Д = \sum_{\nu=1}^{\nu_0} \Delta_\nu, \qquad \overline{\Delta}_\nu = (\xi_\nu(t) \leqslant x \leqslant \xi_{\nu+1}(t),\quad 0 \leqslant t \leqslant T).$$

If $\xi_\nu'(t) \equiv 0$ then $x = \xi_\nu(t)$ is a straight line parallel to the t-axis in the (x, t) plane ("a fixed discontinuity").

We shall assume everwhere that the problem (1)-(5) has a unique solution $u = u(x, t)$, which is continuous in $\overline{Д}$ and possesses whatever derivatives we need in the course of our work. Similar assumptions are made with respect to the other problems considered below.

2. *Homogeneous difference schemes.* Let $\overline{\omega}_h = \{x_i, 0 \leqslant i \leqslant N\}$ be an arbitrary difference net on the segment $0 \leqslant x \leqslant 1$, let $\overline{\omega}_\tau = \{t_j, 0 \leqslant j \leqslant K\}$ be the net on the segment $0 \leqslant t \leqslant T$, where $t_0 = 0$, $t_K = T$; $\omega_h = \{x_i,\ 0 < i < N\}$, $\omega_\tau = \{t_j, 0 < j \leqslant K\}$. Let $\overline{\Omega} = \overline{\omega}_h \times \overline{\omega}_\tau = \{(x_i, t_j) \in \overline{Д}\}$ be the space-time net, and $\Omega = \omega_h \times \omega_\tau$ the set of internal nodes of the net $\overline{\Omega}$. The steps $h_i = x_i - x_{i-1} > 0$, $0 < i \leqslant N$ $\tau_j = t_j - t_{j-1} > 0$, $0 < j \leqslant K$, of the nets $\overline{\omega}_h$ and $\overline{\omega}_\tau$ are arbitrary and satisfy only the normalization conditions

$$\sum_{i=1}^{N} h_i = 1, \qquad \sum_{j=1}^{K} \tau_j = T. \tag{6}$$

If $\tau_j - \tau_{j-1} = O(\tau_j\tau_{j-1})$ or $\tau_j - \tau_{j-1} = O(\tau_j^2)$, then we shall call the net $\overline{\omega}_\tau$ a quasi-uniform net.

For net functions we shall, as a rule, make use of notation without indices, putting

$$h = h_i, \quad h_+ = h_{i+1}, \quad \hbar = 0.5\,(h + h_+), \quad \tau = \tau_{j+1}, \quad \check{\tau} = \tau_j,$$
$$y = y\,(x,\,t) = y_i^{j+1} = y^{j+1} = y_i, \qquad y_{\bar{t}} = (y - \check{y})/\tau.$$

The notation $y_{\bar{x}}, y_x, y_{\hat{x}}, y_{\check{x}}$ was introduced in Point 1, Section 1.

We shall introduce any other notation as it is needed.

We put the problem (1)-(5) in correspondence with the six-point homogeneous difference scheme (cf. [2]):

$$\left.\begin{aligned} &\rho^{(\alpha)} y_{\bar{t}} = (L_h^{(k,\,q,\,f)} y)^{(\alpha)}, \quad 0 \leqslant \alpha \leqslant 1, \qquad (x,\,t) \in \Omega, \\ &y_0 = y\,(0,\,t) = u_1\,(t), \quad y_N = y\,(1,\,t) = u_2\,(t), \quad t \in \omega_\tau; \\ &y\,(x,\,0) = u_0\,(x), \quad x \in \omega_h, \end{aligned}\right\} \tag{7}$$

where $\rho^{(\alpha)} = \alpha\rho + (1 - \alpha)\,\check{\rho}$ and $L_{h}^{(k,\,q,\,f)} y = (ay_{\bar{x}})_{\hat{x}} - dy + \varphi$ is the scheme defined in Section 1.

The coefficients a, d, φ, ρ are given by the formulae

$$a = a\,(x,\,t) = A\,\,[k\,(x + sh,\,t)],$$
$$d = d\,(x,\,t) = \frac{h}{\hbar} D\,\,[q\,(x + sh,\,t)\,\eta_0^-(s)] + \frac{h_+}{\hbar} D\,\,[q\,(x + sh_+,\,t)\,\eta_0^+(s)],$$
$$\varphi = \varphi\,(x,\,t) = \frac{h}{\hbar} F\,\,[f\,(x + sh,\,t)\,\eta_0^-\,(s)] + \frac{h_+}{\hbar} F\,\,[f\,(x + sh_+,\,t)\,\eta_0^+\,(s)],$$
$$\rho = \rho\,(x,\,t) = \frac{h}{\hbar} R\,\,[c\,(x + sh,\,t)\,\eta_0^-(s)] + \frac{h_+}{\hbar} R\,\,[c\,(x + sh_+,\,t)\,\eta_0^+(s)].$$

The pattern functionals A, D, F and R satisfy all the requirements (1)-(4) given in Point 1 of Section 1, and R possesses the same properties as D and F, so that

$$R\,\,[1] = 1, \quad R\,\,[s] = 0, \quad R\,\,[\eta_0^-(s)] = \tfrac{1}{2}, \quad R\,\,[\pi_0\,(s)] = 0.$$

In addition we shall assume everywhere that $A[\bar{k}(s)]$ has a third differential.

From (4) and the properties of the pattern functionals it follows that

$$0 < c_1 \leqslant a, \qquad 0 < c_2 \leqslant \rho, \qquad 0 \leqslant d. \tag{8}$$

3. *Approximation error on a non-uniform net*. Let $u = u(x,\ t)$ be the

solution of the problem (1)-(5), and y the solution of the problem (7). The net function $z = y - u$ is determined by the conditions:

$$\left.\begin{aligned} \rho^{(\alpha)} z_{\bar{t}} = (\Lambda z)^{(\alpha)} + \Psi, \qquad \Lambda z = (a z_{\bar{x}})_{\hat{x}} - dz, \\ z_0 = z_N = 0, \qquad z(x, 0) = 0, \end{aligned}\right\} \tag{9}$$

$$\Psi = (L_h^{(k, q, f)} u)^{(\alpha)} - \rho^{(\alpha)} u_{\bar{t}},$$

where Ψ is the approximation error of the scheme (7) in the class of solutions $u = u(x, t)$ of the differential equation (1).

We shall assume everywhere that the conditions ensuring the maximum order of approximation of the scheme (7) on a uniform net (conditions A) are satisfied. These conditions are satisfied either in the whole region $\bar{Д}$ or in each of the regions $\bar{\Delta}_\nu$ (conditions A_ν) if the coefficients of the differential equation are discontinuous (cf. [4]).

Let us consider some node $(x, t) \in \Omega$. Suppose that in the neighbourhood of this node conditions A are satisfied. Using equation (1) we can write

$$\Psi = (L_h^{(k, q, f)} u - L^{(k, q, f)} u)^{(\alpha)} - \left[\rho^{(\alpha)} u_{\bar{t}} - \left(c \frac{\partial u}{\partial t}\right)^{(\alpha)}\right].$$

On the uniform net $\Omega(\tau = T/K = \text{const.},\ h = 1/N = \text{const.})$, because of conditions A, we have

$$\Psi = O(h^2) + O(\tau^{m_\alpha}), \qquad m_\alpha = \begin{cases} 2, & \alpha = 0.5, \\ 1, & \alpha \neq 0.5. \end{cases}$$

We put Ψ in the form

$$\Psi = \psi^{(\alpha)} + \bar{\psi}, \qquad \psi = L_h^{(k, q, f)} u - L^{(k, q, f)} u - (\rho - c) \frac{\partial u}{\partial t},$$

$$\bar{\psi} = -\alpha\rho\left(u_{\bar{t}} - \frac{\partial u}{\partial t}\right) - (1 - \alpha)\check{\rho}\left(u_{\bar{t}} - \frac{\partial \check{u}}{\partial t}\right).$$

Using the representation (1.25) we find

$$\psi = \overset{**}{\mu}_{\hat{x}} + O(h^2) + O(h_+^2),$$

$$\overset{**}{\mu} = h^2 \overline{\overset{**}{\mu}_0}, \qquad \overset{**}{\mu}_0 = \overset{*}{\mu}_0 - \frac{\partial c}{\partial x} \frac{\partial u}{\partial t} R\, [s\eta_0^-(s)],$$

where μ_0^* is given by formula (1.26).

It is not difficult to see that $\bar{\psi} = O(\tau^{m_\alpha})$. Thus we have proved the following lemma.

Lemma 1. The approximation error of the scheme (7) can be put in the form

$$\left.\begin{aligned} &\Psi = \mu_{\hat{x}}^{(\alpha)} + \psi^*, \qquad \psi^* = O(\tau^{m_\alpha}) + O(h^2) + O(h_+^2), \\ &\mu = \bar{\mu}_0 h^2, \qquad \bar{\mu}_0 = \mu_0(x - 0.5h, t), \qquad \mu_0 = \beta u' + \frac{1}{8}(ku')'' + \\ &+ q'uD\,[s\eta_0^-(s)] - f'F\,[s\eta_0^-(s)] + c'\frac{\partial u}{\partial t}R\,[s\eta_0^-(s)], \end{aligned}\right\} \quad (10)$$

$$\beta = \frac{k''}{2}\left(A_1\,[s^2] - \frac{1}{4}\right) + \frac{(k')^2}{k}A_2\,[s].$$

(The dash denotes differentiation with respect to x.)

Note 1. It follows from conditions A and equation (1) that $\partial u/\partial t$ satisfies the Lipschitz condition in x.

Note 2. It is clear from (10) that on a non-uniform net the scheme (7) has first order approximation with respect to x whatever the order of smoothness of the function $u = u(x, t)$ and the coefficients of equation (1) may be.

Consider the case when k, q, f, c have a finite number of discontinuities on the straight lines $x = \xi_\nu = \text{const.}(t)$ $(\nu = 1, 2, \ldots, \nu_0)$ parallel to the t-axis in the (x, t) plane (fixed discontinuities). Since the lines of discontinuity are fixed, it is always possible to change the net ω_h in the neighbourhood of the points $x = \xi_\nu \in (0, 1)$ so that the points $x = \xi_\nu$ will be nodes of the net ω_h, i.e. $\xi_\nu = x_{n_\nu}$, $0 < n_\nu < N$ (n_ν does not depend on t). The sequence of nets $\bar{\omega}_h$ constructed in this way, depending on the choice of the coefficients k, q, f, c, is denoted by $\bar{\omega}_h(k)$, and the corresponding sequence of space-time nets by $\bar{\Omega}(k) = \bar{\omega}_h(k) \times \bar{\omega}_\tau$.

Let $\xi = x_n$ be one of the points of discontinuity at which $k_1 \neq k_r$, $q_1 \neq q_r$, $f_1 \neq f_r$, $c_1 \neq c_r$. At this point we have the coupling conditions

$$[u] = u_r - u_1 = 0, \quad \left[k\frac{\partial u}{\partial x}\right] = 0 \quad \text{for} \quad x = \xi \in \omega_h. \qquad (11)$$

By analogy with Section 1 we define the approximation error of the scheme (7) at the point $\xi \in \omega_h$ as:

$$\Psi = (L_h^{(k,q,f)}u - \widetilde{L^{(k,q,f)}u})^{(\alpha)} - \left[\rho^{(\alpha)}u_{\bar{t}} - \widetilde{\left(c\frac{\partial u}{\partial t}\right)}^{(\alpha)}\right]. \qquad (12)$$

Since the discontinuity is fixed, $[\partial u/\partial t] = 0$ when $x = \xi$.

Using the results of Point 3, Section 1 together with the argument used in the proof of Lemma 1 we can see that formulae (10) remain in force at the point of discontinuity of the coefficients of equation (1), $\xi \in \omega_h$, and we therefore have the following lemma.

Lemma 2. If the coefficients k, q, f, c have a finite number of fixed discontinuities and in each of the regions $\bar{\Delta}_\nu$ $(\nu = 0, 1, \ldots, \nu_0)$ conditions A are satisfied, then on the sequence of nets $\bar{\Omega}(k)$ the approximation error of any of the schemes (7) is given by formulae (10) at all nodes $(x, t) \in \Omega(k)$.

If k, q, f and c have discontinuities of the first kind on the curves $x = \xi_\nu(t) = x_{n_\nu} + \theta_\nu h_{n_\nu+1}$, $0 < \theta_\nu < 1$, then (10) is not valid at the nodes (x_{n_ν}, t) and $(x_{n_\nu+1}, t)$.

4. *A linear equation of general form.* Let us now consider a linear parabolic equation of general form

$$c(x, t)\frac{\partial u}{\partial t} = L^{(k, q, f)}u + r(x, t)\frac{\partial u}{\partial x}, \tag{13}$$

$u(0, t) = u_1(t)$, $u(1, t) = u_2(t)$, $u(x, 0) = u_0(x)$.

In order to write down a homogeneous difference scheme on a non-uniform net for this equation we must find a suitable difference approximation for the term $r\frac{\partial u}{\partial x}$, which is also appropriate for the case of discontinuous coefficients, and, in particular, on the sequence of nets $\bar{\omega}_h(k)$. We shall use one of the following approximations:

$$r\frac{\partial u}{\partial x} \sim \lambda(y) = \begin{cases} b^- y_{\check{x}} + b^+ y_{\hat{x}} = \frac{h}{\hbar} b^- y_{\bar{x}} + \frac{h_+}{\hbar} b^+ y_x, \\ b\,(a^{(+1)} y_x + a y_{\bar{x}}), \end{cases} \tag{14}$$

where

$$y_{\check{x}} = \frac{h}{\hbar} y_{\bar{x}}, \qquad y_{\hat{x}} = \frac{h_+}{\hbar} y_x,$$

$$b^- = \frac{h}{\hbar} B\,[r(x + sh, t)\,\eta_0^-(s)], \qquad b^+ = \frac{h_+}{\hbar} B\,[r(x + sh_+, t)\,\eta_0^+(s)],$$

$$b = \frac{h}{\hbar} B\,[\tilde{r}(x + sh, t)\,\eta_0^-(s)] + \frac{h_+}{\hbar} B\,[\tilde{r}(x + sh_+, t)\,\eta_0^+(s)], \qquad \tilde{r} = r/2k.$$

Here $B[\mu(s)]$ is a linear non-decreasing functional satisfying the conditions

$$B\,[1] = 1, \quad B\,[s] = 0, \quad B\,[\eta_0^-(s)] = \tfrac{1}{2}, \quad B\,[\pi_0(s)] = 0.$$

Let us find the approximation error for each of the expressions for $\lambda(y)$. Let $x \in \omega_h$ be a point of discontinuity of $r(x, t)$ and $k(x, t)$. Taking $\lambda_1(u) = (hb^- u_{\bar{x}} + h_+ b^+ u_x)/\hbar$, simple calculations give

$$\lambda_1(u) = \frac{1}{2\hbar}(r_l u'_l h + r_r u'_r h_+) + (\bar{\mu}_r)_{\hat{x}} + O(h^2) + O(h_+^2),$$

or

$$\lambda_1(u) - \widetilde{ru'} = (\bar{\mu}_r)_{\hat{x}} + O(h^2) + O(h_+^2), \tag{15}$$

where

$$\bar{\mu}_r = \mu_r(x - 0.5h, t), \qquad \mu_r = (ru'' - r'u'B[s\eta_0^-(s)])\, h^2. \tag{16}$$

If $[ru'] = 0$, then $\widetilde{ru'} = ru'$. If $r(x, t)$ and $k(x, t)$ have discontinuities at the point $\xi = x_n + \theta h_{n+1}$ $(0 < \theta < 1)$, then $(\lambda_1(u) - ru')_{x=x_n} = O(1)$ (for $\theta < 0.5$) and at the point $x = x_n$, $\Psi = O(1)$ even for the scheme

$$A^*[\mu] = \left[\int_{-1}^{0} \frac{ds}{\mu(s)}\right]^{-1}, \qquad B^*[\mu] = D^*[\mu] = F^*[\mu] = R^*[\mu] = \int_{-0.5}^{0.5} \mu(s)\, ds.$$

In this case in order to increase the order of accuracy we must use the second expression for $\lambda(y)$ (cf. [4], [5]).

We now write, by analogy with [4], the difference scheme for (13):

$$\begin{gathered} \rho^{(\alpha)} y_{\bar{t}} = (L_h^{(k,q,f)} y + \lambda(y))^{(\alpha)}, \quad 0 \leqslant \alpha \leqslant 1, \\ y_0 = u_1(t), \quad y_N = u_2(t), \quad y(x, 0) = u_0(x). \end{gathered} \tag{17}$$

If conditions A, which guarantee the maximum order of approximation of the scheme (17), are satisfied, then formulae of the form (10) are valid for the approximation error Ψ of the scheme (17):

$$\left.\begin{aligned} &\Psi = \mu_{\hat{x}}^{(\alpha)} + \psi^*, \qquad \psi^* = O(\tau^{m_\alpha}) + O(h^2) + O(h_+^2), \\ &\mu = h^2 \bar{\mu}_0, \ \mu_0 = \mu_0\left(k, k', k'', q', f', (ku')'', c' \frac{\partial u}{\partial t}, ru'', r'u'\right). \end{aligned}\right\} \tag{18}$$

Besides the scheme (17), schemes of the form

$$\rho y_{\bar{t}} = L_h^{(k,q,f)} y^{(\alpha)} + \lambda(y^{(\alpha)}), \quad y_0 = u_1(t), \quad y_N = u_2(t), \quad y(x, 0) = u_0(x), \tag{19}$$

are of interest. The coefficients ρ, a, d, φ, b^-, b^+ (or b) can be expressed in terms of the corresponding coefficients of the differential equation (1) taken at time $\bar{t} = t - 0.5\tau$. The scheme (19) is equivalent to scheme (17) with respect to the order of approximation.

In practice it is sometimes advisable to use the schemes

$$\rho y_{\bar{t}} = (a y_{\bar{x}})_{\hat{x}}^{(\alpha)} + (ly)_{(\beta)}, \qquad 0 \leqslant \alpha \leqslant 1, \ \ 0 \leqslant \beta \leqslant 1, \tag{20}$$

where $ly = -dy + \varphi + \lambda(y)$ and $(ly_{(\beta)}$ is given by one of the formulae:

$$(ly)_{(\beta)} = \beta ly + (1-\beta)\, \check{l}\check{y} = (ly)^{(\beta)},$$
$$(ly)_{(\beta)} = -d(x, \bar{t})\, y^{(\beta)} + \varphi(x, \bar{t}) + b^{-}(x, \bar{t})\, y_{\bar{x}}^{(\beta)} + b^{+}(x, \bar{t})\, y_{x}^{(\beta)}, \ \ \bar{t} = t - 0.5\tau.$$

In particular, the scheme corresponding to $\beta = 0$, in which all the earlier terms are taken on the preceding row, is very convenient in practice. In this case the successive substitution formulae may be used on any net $\bar{\omega}_h$, whatever the sign of d and $b^{\pm}$ may be (cf. [4], Section 2, Point 4).

5. *A quasi-linear equation.* In [5] we studied homogeneous difference schemes on uniform nets for the quasi-linear equations

$$\left.\begin{array}{c} \mathcal{P}u = \dfrac{\partial}{\partial x}\left(k(x, t, u)\dfrac{\partial u}{\partial x}\right) - c(x, t)\dfrac{\partial u}{\partial t} + f\left(x, t, u, \dfrac{\partial u}{\partial x}\right) = 0 \\ (k(x, t, u) \geqslant c_1 > 0, \quad c(x, t) \geqslant c_2 > 0), \\ u(0, t) = u_1(t), \quad u(1, t) = u_2(t), \qquad u(x, 0) = u_0(x). \end{array}\right\} \tag{21}$$

We shall assume that the derivatives of the functions $k(x, t, u)$ and $f(x, t, u, p)$ with respect to the arguments u and p satisfy the same conditions as in [5]. When the "heat conduction coefficient" $k = k(x, t, u)$ depends on u special difficulties arise. We succeeded in proving uniform convergence in [5] only for four-point implicit schemes, the estimate of their order of accuracy being considerably cruder than in the case $k = k(x, t)$. In this article we extend the class of converging schemes and obtain new estimates of the rate of convergence which are, in particular, more exact than those of [5] in the case of uniform nets.

We shall consider the following class of schemes:

$$\rho y_{\bar{t}} = (a_{(\alpha)} y_{\bar{x}})_{\hat{x}} + \varphi_{(\beta)}, \qquad 0 \leqslant \alpha \leqslant 1, \ \ 0 \leqslant \beta \leqslant 1, \tag{22}$$
$$y_0 = u_1(t), \quad y_N = u_2(t), \quad y(x, 0) = u_0(x),$$

where $a_{(\alpha)}$ and $\varphi_{(\beta)}$ are found from one of the corresponding formulae:

$$a_{(\alpha)} = \alpha a(x, t, y^*) + (1-\alpha)\, a(x, \check{t}, \check{y}^*) = a^{(\alpha)}(x, t, y^*),$$
$$y^* = 0.5\,(y + y^{(-1)}), \quad \check{t} = t - \tau,$$
$$a_{(\alpha)} = a(x, t^{(\alpha)}, (y^*)^{(\alpha)}), \qquad (y^*)^{(\alpha)} = \alpha y^* + (1-\alpha)\, \check{y}^*,$$

$$\varphi_{(\beta)} = \beta\varphi(x, t, y, \lambda(y)) + (1-\beta)\varphi(x, \check{t}, \check{y}, \check{\lambda}(\check{y})) = \varphi^{(\beta)}(x, t, y, \lambda(y)),$$

$$\varphi_{(\beta)} = \varphi(x, t^{(\beta)}, y^{(\beta)}, \lambda^{(\beta)}(y)), \quad \lambda(y) = 0.5(y_{\hat{x}} + y_{\check{x}}) = y_{\overset{\circ}{x}} = (y^{(+1)} - y^{(-1)})/2\hbar,$$

$$a(x, t, u) = A\,[k(x + sh, t, u)], \qquad -1 \leqslant s \leqslant 0,$$

$$\varphi(x, t, u, p) = \frac{h}{\hbar} F\,[f(x + sh, t, u, p)\,\eta_0^-(s)] + {}$$
$$+ \frac{h_+}{\hbar} F\,[f(x + sh_+, t, u, p)\,\eta_0^+(s)],$$

$$\rho = \rho(x, t) = \frac{h}{\hbar} F\,[c(x + sh, t)\,\eta_0^-(s)] + \frac{h_+}{\hbar} F\,[c(x + sh_+, t)\,\eta_0^+(s)].$$

The properties of the functionals A and F are given in Point 1, Section 1; in addition, (as in Point 2, Section 2) we make the requirement that $A[\bar{k}(s)]$ shall have a third differential. It was shown in [5] that the pattern functionals

$$A^*\,[\mu(s)] = \left[\int_{-1}^{0} \frac{ds}{\mu(s)}\right]^{-1}, \qquad F^*\,[\mu(s)] = \int_{-0.5}^{0.5} \mu(s)\,ds$$

ensure a higher order of accuracy in the class of discontinuous coefficients of equation (21). The function f in (21) must be transformed to the form

$$f = f\left(x, t, u, 2k\frac{\partial u}{\partial x}\right).$$

We then have

$$\varphi = \varphi(x, t, y, \lambda(y)), \qquad \lambda(y) = a^{(+1)}y_{\hat{x}} + ay_{\check{x}},$$
$$a(x, t, u) = A^*\,[k(x + sh, t, u)],$$
$$\varphi(x, t, u, p) = F^*\,[f(x + (s + \Delta)\hbar, t, u, p)],$$
$$\rho(x, t) = F^*\,[c(x + (s + \Delta)\hbar, t)], \qquad \Delta = (h_+ - h)/4\hbar.$$

As usual, $u = u(x, t)$ is the solution of the initial problem (21), $y = y(x, t)$ is the solution of problem (22). For the net function $z = y - u$ we obtain the conditions

$$\rho z_{\bar{t}} = (a_{(\alpha)} z_{\bar{x}})_{\hat{x}} + Q(z) + \psi, \qquad z_0 = 0, \qquad z_N = 0, \qquad z(x, 0) = 0, \tag{23}$$
$$a_{(\alpha)} \geqslant c_1 > 0, \qquad \rho \geqslant c_2 > 0,$$

where $Q(z)$ is given by an expression of the form

$$Q(z) = d_1 z + d_2\check{z} + b_{11}z_{\hat{x}} + b_{12}\check{z}_{\hat{x}} + b_{22}z_{\check{x}} + b_{21}\check{z}_{\check{x}} + (g_{11}z)_{\hat{x}} + {} \tag{24}$$
$$+ (g_{12}\check{z})_{\hat{x}} + (g_{22}z)_{\check{x}} + (g_{21}\check{z})_{\check{x}}$$

for some special values of the boundary coefficients d_s, b_{sk}, g_{sk}, s, $k = 1, 2$.

Reasoning by analogy with Point 2 we find the following expression for the approximation error:

$$\psi = \mu_{\hat{x}} + \psi^*, \qquad \psi^* = O(h^2) + O(h_+^2) + O(\tau), \qquad \mu = O(h^2). \quad (25)$$

We shall not write out an explicit expression for μ.

3. *A priori* estimates

To find the order of accuracy of the difference schemes examined in Section 2 we need various *a priori* estimates establishing the stability of the linear equations (2.9) and (2.23) with respect to the right-hand side. Estimates of this kind were obtained in [1], [2], [4], [5]; it was shown how these estimates are modified on non-uniform nets. For the six-point schemes (2.9) we use the estimates of [1], [2] and [3] without any essential alteration. We shall only show that the scheme (2.9) is stable with respect to its right-hand side of any $\|\tau\|_0 = \max \tau_j$, while its stability when $\tau < \tau_0$ is proved in [2].

We shall concentrate on the derivation of a uniform *a priori* estimate for the equation (2.23), special cases of which were studied in [1] and [5]. With the help of an *a priori* estimate in the mean we substantially strengthen Theorem 4 of [5].

1. *Introduction*. We shall distinguish between two cases:

a) $k(x, t)$ bounded:

$$0 < c_1 \leqslant k(x, t) \leqslant c_1', \quad (1)$$

and satisfies the Lipschitz condition in t:

$$|k_{\bar{t}}| \leqslant c_4' \qquad (c_4' = \text{const.} > 0); \quad (2)$$

b) $k(x, t)$ or $k(x, t, u)$ is bounded only.

Lemma 3. If conditions (1) and (2) are satisfied and the pattern functional $A[\bar{k}(s)]$ has a first differential, then for the coefficient $a(x, t) = A[\bar{k}(x + sh, t)]$ we have the inequality

$$|a_{\bar{t}}| \leqslant c_4, \text{ where } c_4 = c_4' \frac{c_1'}{c_1}. \quad (3)$$

We recall that $A[k]$ is a non-decreasing homogeneous functional of the first degree satisfying the normalisation condition $A[1] = 1$.

Let $A_1[k, \mu]$ be the first differential of the functional $A[k]$. We show first that

$$\frac{c_1}{c_1'} \leqslant A_1 [k, 1] \leqslant \frac{c_1'}{c_1}. \tag{4}$$

To do this we use the identity $A_1 [k, k] = A [k]$ (see [7], Section 1) and the inequalities

$$c_1 A_1 [k, 1] = A_1 [k, c_1] \leqslant A_1 [k, k] = A [k] \leqslant c_1',$$
$$c_1 \leqslant A_1 [k, k] \leqslant A_1 [k, c_1'] = c_1' A_1 [k, 1].$$

From the theorem about the mean value we know that the difference $a - \check{a}$ is equal to

$$a - \check{a} = A [k (x + sh, t)] - A [k (x + sh, \check{t})] =$$
$$= A [k] - A [\check{k}] = A_1 [\check{k} + \theta \tau k_{\bar{t}}, \tau k_{\bar{t}}], \qquad 0 < \theta < 1.$$

It follows from this and from (4) that

$$|a_{\bar{t}}| \leqslant c_4' A_1 [\check{k} + \theta \tau k_{\bar{t}}, 1] \leqslant c_4' \frac{c_1'}{c_1} = c_4.$$

Note. If $A[k]$ is a linear functional, then $|a_{\bar{t}}| \leqslant c_4$, i.e. $c_4 = c_4'$.

Lemma 4. Let g_j and ρ_j be functions which are given on an arbitrary non-uniform net $\bar{\omega}_\tau$ with step $\tau_j = t_j - t_{j-1} > 0$; $j = 1, 2, \ldots$ If ρ_j is a non-negative non-decreasing function ($\rho_{j+1} \geqslant \rho_j$), then the inequality

$$g_{j+1} \leqslant c_0 \sum_{j'=2}^{j+1} \tau_{j'} g_{j'-1} + \rho_{j+1} \qquad (c_0 = \text{const}. > 0) \tag{5}$$

implies

$$g_{j+1} \leqslant e^{c_0 t_{j+1}'} \rho_{j+1}, \qquad j = 0, 1, \ldots. \tag{6}$$

If ρ_j is an arbitrary non-negative function, it follows from (5) that

$$g_{j+1} \leqslant \rho_{j+1} + c_0 e^{c_0 t_{j+1}'} \sum_{j'=2}^{j+1} \tau_{j'} \rho_{j'-1}, \qquad t_{j+1}' = t_{j+1} - \tau_1. \tag{7}$$

Let us first obtain the estimate (6). We shall look for g_j in the form

$$g_j = \sum_{k=0}^{\infty} g_j^{(k)}, \tag{8}$$

where

$$g_j^{(0)} \leqslant \rho_j, \qquad g_j^{(k)} \leqslant c_0 \sum_{j'=2}^{j+1} \tau_{j'} g_{j'-1}^{(k-1)}, \qquad k = 1, 2, \ldots .$$

Using the fact that

$$\sum_{j'=2}^{j+1} \tau_{j'} f(t_{j'-1}) \leqslant \int_{\tau_1}^{t_{j+1}} f(t)\, dt, \text{ if } f(t'') \geqslant f(t') \text{ for } t'' \geqslant t',$$

and the monotonicity of ρ_j, we find

$$g_{j+1}^{(k)} \leqslant \frac{(c_0 t'_{j+1})^k}{k!} \rho_{j+1}, \qquad t'_{j+1} = t_{j+1} - \tau_1.$$

From this and from (8) we have (6).

Let ρ_j be an arbitrary non-negative function. Putting $g_{j+1} = \rho_{j+1} + v_{j+1}$, we obtain

$$v_{j+1} \leqslant c_0 \sum_{j'=2}^{j+1} \tau_{j'} v_{j'-1} + \rho^*_{j+1}, \qquad \rho^*_{j+1} = \sum_{j'=2}^{j+1} \tau_{j'} \rho_{j'-1} \geqslant 0,$$

where ρ^*_j is a non-decreasing function. Then, using estimate (6) for v_j we obtain (7).

Corollary. If the net $\bar{\omega}_\tau$ is uniform, i.e. $\tau_j = \tau = \text{const.}$ than $t'_{j+1} = t_{j+1} - \tau_1 = t_j$ and, instead of (6) and (7) we have

$$g_{j+1} \leqslant e^{c_0 t_j} \rho_{j+1}, \tag{6'}$$

$$g_{j+1} \leqslant \rho_{j+1} + c_0 e^{c_0 t_j}\, \tau \sum_{j'=1}^{j} \rho_{j'} . \tag{7'}$$

Lemma 4a. Suppose g_j and f_j are given on the net $\bar{\omega}_\tau = \{t_j,\ j = 0, 1, \ldots, K\}$. If f_j is an arbitrary non-negative function, then

$$g_{j'} \leqslant (1 + c_0 \tau_{j'})\, g_{j'-1} + \tau_{j'} f_{j'}, \qquad j' = 1, 2, \ldots, \tag{9}$$

implies that

$$g_{j+1} \leqslant e^{c_0 t'_{j+1}} \rho_{j+1}, \text{ where } \rho_j = (1 + c_0 \tau_1)\, g_0 + \sum_{j'=1}^{j} \tau_{j'} f_{j'}.$$

To prove this lemma we need only sum (9) with respect to $j' = 1, 2, \ldots, j+1$

and use estimate (6) of Lemma 4.

By analogy with [6] we introduce notation for the sums and norms of net functions. Let v and ψ be arbitrary functions given on $\bar{\omega}_h$. We put

$$(v, \psi) = \sum_{i=1}^{N-1} v_i\psi_i h_i, \qquad (v, \psi)^+ = \sum_{i=1}^{N-1} v_i\psi_i h_{i+1}, \qquad (v, \psi)^* = \sum_{i=0}^{N-1} v_i\psi_i \hbar_i,$$

$$(v, \psi] = \sum_{i=1}^{N} v_i\psi_i h_i, \qquad [v, \psi) = \sum_{i=0}^{N-1} v_i\psi_i h_{i+1},$$

$$\|\psi\|_0 = \max_{x_i \in \bar{\omega}_h} |\psi_i|, \qquad \|\psi\|_\sigma = (1, |\psi|^\sigma)^{1/\sigma}, \qquad \|\psi\|_{\sigma*} = (1, |\psi|^\sigma)^{*1/\sigma}, \qquad \sigma = 1, 2.$$

If the function v is defined on $\omega_h^+ = \{x_i, 0 < i \leqslant N\}$, then

$$\|v\|_\sigma = (1, |v|^\sigma]^{1/\sigma}.$$

Thus, for example, if the function v is given for all $x \in \bar{\omega}_h$, then $v_{\bar{x}}$ is defined for $x \in \omega_h^+$ and so

$$\|v_{\bar{x}}\|_2 = (1, v_{\bar{x}}^2]^{1/2} \qquad (v_{\bar{x}}^2 = (v_{\bar{x}})^2).$$

It is not difficult to see that

$$\|v\|_2 \leqslant \sqrt{2}\, \|v\|_{2*}.$$

We shall also use the norms

$$\|\psi\|_3 = \|\mu\|_2, \qquad \|\psi\|_{3*} = \|\mu\|_1, \tag{10}$$

where μ is the net function defined by the conditions

$$\psi = \mu_{\hat{x}}, \qquad \mu_N = 0 \quad \text{or } \mu_1 = 0.$$

Let us give some difference identities which we shall use later:

$$\left.\begin{aligned} (v, y_{\hat{x}})^* &= (v, y_x)^+ = -(y, v_{\bar{x}}] + (yv)_N - y_0 v_1, \\ (v, y_{\check{x}})^* &= (v, y_{\bar{x}}) = -[y, v_x) + y_{N-1}v_N - (yv)_0, \end{aligned}\right\} \tag{11}$$

$$(v, (ay_{\bar{x}})_{\hat{x}})^* = (v, (ay_{\bar{x}})_x)^+ = -(a, y_{\bar{x}}v_{\bar{x}}] + (ay_{\bar{x}}v)_N - (a^{(+1)}y_x v)_0, \tag{12}$$

$$(v, (ay_{\bar{x}})_{\hat{x}})^* = ((av_{\bar{x}})_{\hat{x}}, y)^* + (a\,(vy_{\bar{x}} - yv_{\bar{x}}))_N - (a^{(+1)}\,(vy_x - yv_x))_0. \tag{13}$$

If y and v satisfy the homogeneous conditions

$$y_0 = y_N = 0, \qquad v_0 = v_N = 0,$$

all the permutations in formulae (11)-(13) are equal to zero. We also need the inequalities

$$\|v\|_0 \leqslant \|v_{\bar{x}}\|_2, \qquad \|v\|_0 \leqslant \frac{1}{\sqrt{c_1}}\|\sqrt{a}\, v_{\bar{x}}\|_2 \qquad \text{for } v_0 = v_N = 0, \quad a \geqslant c_1 > 0. \tag{14}$$

$$|(y, v)| \leqslant (1, |y|^p)^{1/p} \qquad (1, |v|^q)^{1/q}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad p > 0, \quad q > 0. \tag{15}$$

$$\prod_{k=1}^{m} x_k^{\mu_k} \leqslant \sum_{k=1}^{m} \mu_k x_k, \text{ where } \mu_k \geqslant 0, \qquad x_k \geqslant 0, \qquad \sum_{k=1}^{m} \mu_k = 1. \tag{16}$$

2. *Stability of the scheme (2.9) with respect to its right-hand side.* In [1], [2], [3] we considered the scheme

$$\left.\begin{aligned} \rho z_{\bar{t}} = (\Lambda z)^{(\alpha)} + \Psi, \qquad \Lambda z = (a z_{\bar{x}})_{\hat{x}} - dz, \\ z_0 = 0, \qquad z_N = 0, \qquad z(x, 0) = 0, \\ a \geqslant c_1 > 0, \qquad \rho \geqslant c_2 > 0, \qquad 0 \leqslant d \leqslant c_3, \end{aligned}\right\} \tag{17}$$

and showed that this scheme is uniformly stable with respect to its right-hand side if $0.5 \leqslant \alpha \leqslant 1$ and

$$|a_{\bar{t}}| \leqslant c_4, \tag{18}$$

$$\tau \leqslant \tau_0, \tag{19}$$

where τ_0 is a positive constant which depends only on $c_1, \ldots, c_4$.

Let us show that condition (19) is superfluous.

Theorem 2. The solution of problem (17) satisfies the inequality

$$\|z(x, t)\|_0 \leqslant M\left(\sum_{t_{j'}=t_1}^{t} \tau_{j'} \|\Psi(x, t_{j'})\|_{2^*}^2\right)^{1/2}, \tag{20}$$

if

1) $|a_{\bar{t}}| \leqslant c_4, \qquad |\rho_{\bar{t}}| \leqslant c_5, \qquad 0.5 \leqslant \alpha \leqslant 1,$

2) $\bar{\omega}_h$ is an arbitrary net, $\bar{\omega}_\tau$ a quasi-uniform net (M is a positive constant which depends only on $c_1, \ldots, c_5$). If $\bar{\omega}_\tau$ is an arbitrary non-uniform net, then estimate (20) is valid with the condition that $\|\tau\|_0 = \max \tau_j \leqslant \tau_0$, where $\tau_0 > 0$ is a constant which depends only on the constants $c_1, \ldots, c_4$.

We give the proof in two stages: we first obtain an estimate in the mean (by analogy with [2] and [5]), and then we use it to obtain a uniform estimate.

1. For simplicity in the calculations we give the proof for $\rho = 1$, τ = const. We rewrite equation (17) in the form

$$z - \alpha\tau\Lambda z = \check{z} + (1-\alpha)\,\tau\check{\Lambda}\check{z} + \tau\Psi.$$

We square both sides, multiply by $\hbar$ and sum with respect to all $x \in \omega_h$. Using the first Green formula (12) and the inequality

$$|(\psi, v)^*| \leqslant \frac{1}{2c_0}\|\psi\|_{2*}^2 + \frac{c_0}{2}\|v\|_{2*}^2,$$

where c_0 is an arbitrary positive constant, we obtain

$$H + 2\tau\,(\alpha I + (1-\alpha)\,\check{I}) \leqslant (1 + c_0\tau)\,\check{H} + \tau\,(\tau + 1/c_0)\,\|\Psi\|_{2*}^2, \tag{21}$$

where

$$H = \|z\|_{2*}^2 + \frac{\tau^2}{4}\|\Lambda z\|_{2*}^2, \qquad I = (a, z_{\bar{x}}^2] + (d, z^2)\,.$$

Putting $c_0 = 1/t_{j+1}$ and using Lemma 4 we find

$$\|z(x, t_{j+1})\|_{2*}^2 + \frac{\tau_{j+1}^2}{4}\|(\Lambda z)^{j+1}\|_{2*}^2 + \sum_{j'=1}^{j+1}\tau_{j'}I^{j'} \leqslant e^{t_{j+1}}\sum_{j'=1}^{j+1}\tau_{j'}\|\Psi^{j'}\|_{2*}^2. \tag{22}$$

In the general case, when $\rho = \rho(x, t)$ and $\bar{\omega}_\tau$ is a quasi-uniform net $(|\tau_{\bar{t}}| \leqslant m^*\check{\tau},\ m^* > 0)$, instead of (22) we have

$$\|(\sqrt{\rho}z)^{j+1}\|_{2*}^2 + \frac{1}{4}\tau_{j+1}^2\left\|\left(\frac{\Lambda z}{\sqrt{\rho}}\right)^{j+1}\right\|_{2*}^2 + \sum_{j'=1}^{j+1}\tau_{j'}I^{j'} \leqslant M\sum_{j'=1}^{j+1}\tau_{j'}\|\Psi^{j'}\|_{2*}^2, \tag{23}$$

where M is a positive constant depending only on c_2, c_6, m^* and T. From (23) we have, in particular

$$\sum_{j'=1}^{j+1}\tau_{j'}I^{j'} \leqslant M\sum_{j'=1}^{j+1}\tau_{j'}\|\Psi^{j'}\|_{2*}^2. \tag{24}$$

In deriving (24) we have not used the condition $d \leqslant c_3$.

2. Let us multiply (17) by $\tau z_{\bar{t}}\hbar$ and sum for $x \in \omega_h$:

$$\tau\|\sqrt{\rho}z_{\bar{t}}\|_{2*}^2 + \alpha\,(a, z_{\bar{x}}^2] = (1-\alpha)\,(\check{a}, \check{z}_{\bar{x}}^2] - \tau\,(\alpha dz + (1-\alpha)\,\check{d}\check{z}, z_{\bar{t}})^* + \\ + (\alpha a - (1-\alpha)\,\check{a}, z_{\bar{x}}\check{z}_{\bar{x}}] + \tau\,(\Psi, z_{\bar{t}})^*. \tag{25}$$

Using the obvious inequalities

$$(\alpha a-(1-\alpha)\check{a},\ z_{\bar{x}}\check{z}_{\bar{x}}]=(2\alpha-1)(a,\ z_{\bar{x}}\check{z}_{\bar{x}}]+(1-\alpha)\tau(a_{\bar{t}},\ z_{\bar{x}}\check{z}_{\bar{x}}]\leqslant$$
$$\leqslant(\alpha-0.5)\{(a,\ z_{\bar{x}}^2]+(\check{a},\ \check{z}_{\bar{x}}^2]\}+\tau(c_4/2c_1)\{(1-\alpha)(a,\ z_{\bar{x}}^2]+\alpha(\check{a},\ \check{z}_{\bar{x}}^2]\},$$
$$-(\alpha dz+(1-\alpha)d\check{z},\ \tau z_{\bar{t}})^*\leqslant\frac{\tau}{2}\|\sqrt{\rho}z_{\bar{t}}\|_{2^*}^2+$$
$$+\frac{\tau}{2c_2}\sqrt{c_3}\{\alpha(d,\ z^2)^*+(1-\alpha)(\check{d},\ \check{z}^2)^*\},$$
$$\tau(\Psi,\ z_{\bar{t}})^*\leqslant\frac{\tau}{2}\|\sqrt{\rho}z_{\bar{t}}\|_{2^*}^2+\frac{\tau}{2c_2}\|\Psi\|_{2^*}^2,$$

we obtain from (25), for $0.5\leqslant\alpha\leqslant 1$

$$(a,\ z_{\bar{x}}^2]\leqslant(\check{a},\ \check{z}_{\bar{x}}^2]+\tau c^*(I+\check{I})+\frac{\tau}{c_2}\|\Psi\|_{2^*}^2,$$

where

$$I=(a,\ z_{\bar{x}}^2]+(d,\ z^2)^*,\qquad c^*=\max\left(\frac{c_4}{c_1},\ \frac{\sqrt{c_3}}{c_2}\right).$$

This gives

$$(a,\ z_{\bar{x}}^2]^{j+1}\leqslant 2c^*\sum_{j'=1}^{j+1}\tau_{j'}I^{j'}+\frac{1}{c_2}\sum_{j'=1}^{j+1}\tau_{j'}\|\Psi^{j'}\|_{2^*}^2.$$

Then using estimate (24) and inequality (14) we obtain (20).

Note 1. If $z(x,\ 0)\neq 0$, then on the right-hand side of (23) there will be the additional term

$$M\left\{\|\sqrt{\rho}z\|_{2^*}^2+\tau\|\sqrt{a}z_{\bar{x}}\|_{2^*}^2+\tau(d,\ z^2)^*+\frac{\tau^2}{4}\left\|\frac{\Lambda z}{\sqrt{\rho}}\right\|_{2^*}^2\right\}_{t=0},$$

which can be majorised by the expression

$$(M+M'\gamma+M''\gamma^2)\|z(x,\ 0)\|_{2^*}^2,\qquad \gamma=\|\tau_1/h^2\|_0=\max_{\omega_h^+}(\tau_1/h_i^2),$$

where $M=M(c_1,\ \ldots,\ c_5,\ c_2',\ T,\ m^*)$, $M'=M(c_1,\ \ldots,\ c_5,\ c_1',\ c_2',\ T,\ m^*)$, $M''=M(c_1,\ \ldots,\ c_5,\ c_1',\ c_2',\ T,\ m^*)$ are positive constants with

$$c_1'\geqslant a\geqslant c_1>0,\ c_2'\geqslant\rho\geqslant c_2>0\quad(\text{cf. [2]}).$$

In this case we have instead of (20) an estimate of the form

$$\|z(x,\ t_{j+1})\|_0\leqslant M\left\{(1+\gamma)\|z(x,\ 0)\|_0+\left(\sum_{j'=1}^{j+1}\tau_{j'}\|\Psi^{j'}\|_{2^*}^2\right)^{1/2}\right\}.\tag{20'}$$

Note 2. The estimate (24) still holds if we replace the condition $0 \leqslant d \leqslant c_3$ by the conditions

$$0 \leqslant d, \quad |d_{\bar{t}}| \leqslant c_6.$$

Note 3. The second part of the theorem for the case of an arbitrary net $\bar{\omega}_\tau$ can be proved by analogy with [1] if we use Lemma 4a.

4. *Stability of another scheme for the heat conduction equation.* Let us now consider a scheme (2.19) which is also used in practice:

$$\left.\begin{aligned} \rho z_{\bar{t}} = (a z_{\bar{x}}^{(\alpha)})_{\hat{x}} - d z^{(\alpha)} + \Psi = \Lambda z^{(\alpha)} + \Psi,\ 0.5 \leqslant \alpha \leqslant 1, \\ z_0 = z_N = 0, \\ 0 < c_1 \leqslant a, \quad 0 < c_2 \leqslant \rho, \quad 0 \leqslant d, \end{aligned}\right\} \tag{26}$$

where

$$\rho = \rho(x, \bar{t}), \qquad a = a(x, \bar{t}), \qquad d = d(x, \bar{t}), \qquad \bar{t} = t - 0.5\tau.$$

Theorem 3. The difference scheme (26) is uniformly stable with respect to its right-hand side and the initial data on an arbitrary non-uniform net $\bar{\Omega}$:

$$\| z(x, t_{j+1}) \|_{\bar{0}} \leqslant M' \| z(x, 0) \|_{\bar{0}} + M \Big(\sum_{j'=1}^{j+1} \tau_{j'} \| \Psi^{j'} \|_{2^*}^2 \Big)^{1/2}, \tag{27}$$

where

$$\| z \|_{\bar{0}} = \| z \|_0 + \| z_{\bar{x}} \|_2,$$

if the additional conditions

$$| a_{\bar{t}} | \leqslant c_4, \qquad | d_{\bar{t}} | \leqslant c_5, \qquad a(x, 0) \leqslant c_1', \qquad d \leqslant c_3 \tag{28}$$

are satisfied.

For $\alpha = 1$ the theorem follows from [2]. Let $0.5 \leqslant \alpha \leqslant 1$. As usual we write down the integral identity

$$\tau \| \sqrt{\rho} z_{\bar{t}} \|_{2^*}^2 + \alpha \{(a, z_{\bar{x}}^2] + (d, z^2)^*\} = (1 - \alpha) \{(a, \check{z}_{\bar{x}}^2] + (d, \check{z}^2)^*\} +$$
$$+ (2\alpha - 1) \{(a, z_{\bar{x}} \check{z}_{\bar{x}}] + (d, z\check{z})^*\} + \tau (\Psi, z_{\bar{t}})^*.$$

Using the estimates

$$(a, \check{z}_{\bar{x}}^2] \leqslant \left(1 + \frac{\tau c_4}{c_1}\right)(\check{a}, \check{z}_{\bar{x}}^2], \qquad (a, z_{\bar{x}}\check{z}_{\bar{x}}] \leqslant \tfrac{1}{2}(a, z_{\bar{x}}^2] + \frac{1}{2}\left(1 + \frac{\tau c_4}{c_1}\right)(\check{a}, \check{z}_{\bar{x}}^2],$$

$$(d, \check{z}^2)^* = (\check{d}, \check{z}^2)^* + \tau\,(d_{\bar{t}}, \check{z}^2)^* \leqslant (\check{d}, \check{z}^2)^* + \frac{\tau c_5}{2c_1}(\check{a}, \check{z}_{\bar{x}}^2],$$

$$(d, z\check{z}) \leqslant \tfrac{1}{2}(d, z^2)^* + \frac{1}{2}(d, \check{z}^2)^*,$$

we obtain

$$I \leqslant (1 + c^*\tau)\check{I} + \frac{\tau}{c_2}\|\Psi\|_{2^*}^2,$$

$$I^{j+1} \leqslant c^* \sum_{j'=1}^{j+1} \tau_{j'} I^{j'-1} + I^0 + \sum_{j'=1}^{j+1} \frac{\tau_{j'}}{c_2}\|\Psi^{j'}\|_{2^*}^2,$$

where

$$I = (a, z_{\bar{x}}^2] + (d, z^2)^*, \qquad c^* = \max\left(\frac{c_4}{2c_1}, \frac{c_5}{2c_1}\right),$$

$$I^0 = (a(x, 0), z_{\bar{x}}^2(x, 0)] + (d(x, 0), z^2(x, 0))^*.$$

This and Lemma 4a give (27), if $a(x, 0) \leqslant c_1'$, $d(x, 0) \leqslant c_3$.

The difference scheme (2.19) has second order approximation on the uniform net ω_h and on an arbitrary non-uniform net

$$\Psi = \mu_{\hat{x}}^{(\alpha)} + O(h^2) + O(h_+^2) + O(\tau^{m_\alpha}), \qquad \mu = O(h^2). \tag{29}$$

It follows from this that, generally speaking, the scheme (2.19) is not worse than the scheme (2.9) since it is stable for any $\|\tau\|_0$.

5. *An improved a priori estimate for the scheme (2.17).* When studying the order of accuracy on non-uniform nets we find that the norm $\|\Psi\|_{2^*}$ is too crude, even when the coefficients of the differential equation are continuous. Using the method of stationary solutions explained in [3] and estimate (20) it is not difficult, by analogy with [3], to obtain more refined estimates for $\|z\|_0$.

Let us give the result for the most general problem at once:

$$\left.\begin{aligned} \rho z_{\bar{t}} = (az_{\bar{x}})_{\hat{x}}^{(\alpha)} + Q(z) + \Psi, \qquad z_0 = z_N = 0, \qquad z(x, 0) = 0,\\ Q(z) = d_1 z + d_2\check{z} + b_{11} z_{\hat{x}} + b_{12}\check{z}_{\hat{x}} + b_{22} z_{\check{x}} + b_{21}\check{z}_{\check{x}},\\ 0 < c_1 \leqslant a \leqslant c_1', \; 0 \leqslant c_2 \leqslant \rho \leqslant c_2', \; |d_s| \leqslant c_3, \; |a_{\bar{t}}| \leqslant c_4,\\ |b_{sk}\| \leqslant c_5 \quad (s, k = 1, 2). \end{aligned}\right\} \tag{30}$$

Theorem 4. The solution of problem (30) on an arbitrary non-uniform net for $0.5 \leqslant \alpha \leqslant 1$ and for sufficiently small $\|\tau\|_0 \leqslant \tau_0$ has the estimates:

1)

$$\|z(x, t_{j+1})\|_0 + \|z_{\bar{x}}(x, t_{j+1})\|_2 \leqslant M' (\|z(x, 0)\|_0 + \|z_{\bar{x}}(x, 0)\|_2) + M\Big(\sum_{j'=1}^{j+1} \tau_{j'} \|\Psi^{j'}\|_{2^*}^2\Big)^{1/2}; \tag{31}$$

2) if $\Psi = \psi^{(\alpha)}$, $\psi = \mu_{\hat{x}} + \psi^*$, then

$$\|z(x, t_{j+1})\|_0 \leqslant M \Big\{\|\psi(x, 0)\|_3 + \|\psi(x, t_{j+1})\|_{3^*} + \Big[\sum_{j'=1}^{j+1} \tau_{j'} (\|\psi^{j'}\|_{3^*}^2 + \|\psi_{\bar{t}}^{j'}\|_{3^*}^2)\Big]^{1/2}\Big\} + \overline{M}\Big[\sum_{j'=1}^{j+1} \tau_{j'} (\|\psi^{j'}\|_3^2 + \|\psi^{j'-1}\|_3^2)\Big]^{1/2}, \tag{32}$$

where

$$\|\psi\|_3 = \|\mu\|_2 + \|\psi^*\|_3, \qquad \|\psi\|_{3^*} = \|\mu\|_1 + \|\psi^*\|_{3^*},$$

$\overline{M} = 0$ for a) $Q(z) = (b_1 z_{\hat{x}} + b_2 z_{\check{x}})^{(\alpha)} + d_1 z + d_2 \check{z}$ u b) $|(b_s)_{\bar{t}}| \leqslant c_6$, $s = 1, 2$.

To prove Theorem 4 we have to make a few alterations in the reasoning given in [1] and [3].

6. *Energy identity of the n-th rank*. All the points of Section 3 which follow are concerned with the derivation of new *a priori* estimates for the solution of problem (2.23), i.e. of the problem

$$\rho z_{\bar{t}} = (a z_{\bar{x}})_{\hat{x}} + Q(z) + \psi, \quad z_0 = z_N = 0, \quad z(x, 0) = 0 \tag{33}$$

where $Q(z)$ is given by formula (2.24). We shall assume that the coefficients a, ρ, d_s, b_{sk}, g_{sk} (s, k = 1, 2) are bounded:

$$0 < c_1 \leqslant a, \quad 0 < c_2 \leqslant \rho, \quad \underset{s,\ k = 1,\ 2,}{|d_s| \leqslant c_3,} \quad |b_{sk}| \leqslant c_4, \quad |g_{sk}| \leqslant c_5, \tag{34}$$

and $\rho = \rho(x, t)$, in addition, satisfies the Lipschitz condition with respect to t

$$|\rho_{\bar{t}}| \leqslant c_6 \tag{35}$$

(c_1, ..., c_6 are positive constants which do not depend on the net).

In [5] we considered a special case of equation (33), for $d_2 = b_{12} = b_{21} = g_{21} = g_{12} = 0$ and with boundary conditions of the general form

$$a^{(+1)} z_x = \mathscr{E}_1 z_{\bar{t}} + \sigma_1 z - \nu_1 \text{ for } x = 0, \quad -a z_{\bar{x}} = \mathscr{E}_2 z_{\bar{t}} + \sigma_2 z - \nu_2 \text{ for } x = 1. \tag{36}$$

Although our argument is given for the conditions of the first kind $z_0 = z_N = 0$ the final results are valid for conditions (36) also provided that $\mathscr{E}_s$, σ_s $(s = 1, 2)$ satisfy the requirements given in [6]. It is sufficient here to replace $\|\psi\|_3$ by $\|\psi\|_5$, where

$$\|\psi\|_5 = \|\psi\|_3 + |(\psi, 1)^*| + |\nu_1| + |\nu_2|.$$

We shall assume that $\bar{\omega}_h$ is an arbitrary non-uniform net, and $\omega_\tau = \omega_\tau^*$ is a non-uniform net whose steps satisfy the condition

$$\tau \leqslant m^* \check{\tau}, \tag{37}$$

where m^* is some positive constant. We denote the net corresponding to Ω by $\Omega^* = \omega_h \times \omega_\tau^*$.

Introducing the new function v:

$$z = v w, \quad w_{\bar{t}} = \bar{M} \check{w}, \quad w(x, 0) = 1, \quad \text{or} \quad w = (1 + \bar{M}\tau)\check{w}, \tag{38}$$

where $\bar{M} > 0$ is an arbitrary constant we obtain

$$\left.\begin{aligned}
&\bar{\rho} v_{\bar{t}} - (a v_{\bar{x}})_x + d v = \Psi = \bar{\psi} + Q_1(v), \ v_0 = v_N = 0, \ v(x, 0) = 0, \\
&Q_1(v) = \bar{d}_2 \check{v} + b_{11} v_{\hat{x}} + b_{22} v_{\check{x}} + \bar{b}_{12} \check{v}_{\hat{x}} + \bar{b}_{21} \check{v}_{\check{x}} + (g_{11} v)_{\hat{x}} + \\
&\qquad + (g_{22} v)_{\check{x}} + (\bar{g}_{12} \check{v})_{\hat{x}} + (\bar{g}_{21} \check{v})_{\check{x}}, \\
&\bar{\rho} = \rho\gamma, \ \bar{d} = \rho\gamma\bar{M} - d_1, \ \bar{d}_2 = d_2\gamma, \ \bar{b}_{sk} = b_{sk}\gamma, \ \bar{g}_{sk} = g_{sk}\gamma, \\
&\bar{\psi} = \psi/w, \qquad \gamma = (1 + \bar{M}\tau)^{-1}.
\end{aligned}\right\} \tag{39}$$

Let us write down the equation of the n-th rank, i.e. the equation for the function $\overset{n}{v} = v^{2^n}$, $n = 1, 2, \ldots$ To do this we have to multiply (39) by $v x v^2 \ldots v^{2^{n-1}} = 2^n v^{\alpha_n}$, where $\alpha_n = 2^n - 1$; and use the identity

$$2v\,(a v_{\bar{x}})_{\hat{x}} = (a\,(v^2)_{\bar{x}})_{\hat{x}} - \frac{1}{\hbar}\,[h a v_{\bar{x}}^2 + h_+ a^{(+1)} v_x^2].$$

We then obtain (cf. [1] and [5]):

$$\rho \overset{n}{v}_{\bar t} - (a \overset{n}{v}_{\bar x})_{\hat x} + \sum_{k=0}^{n-1} 2^{n-k-1} \left\{ \frac{h}{\hbar} a (\overset{k}{v}_{\bar x})^2 + \frac{h_+}{\hbar} a^{(+1)} (\overset{k}{v}_x)^2 + \tau \bar\rho (\overset{k}{v}_{\bar t})^2 \right\} \times$$
$$\times\, v^{\alpha_n - \alpha_{k+1}} + 2^n d \overset{n}{v} = 2^n v^{\alpha_n} \Psi.$$

Multiplying (39) by $\hbar$ and summing for $x \in \omega_h$, remembering that

$$\left(\frac{h}{\hbar} a\left(\overset{k}{v}_{\bar x}\right)^2 + \frac{h_+}{\hbar} a^{(+1)} \left(\overset{k}{v}_x\right)^2, v^{\alpha_n - \alpha_{k+1}}\right)^* = \left(a \left(\overset{k}{v}_{\bar x}\right)^2, v^{\alpha_n - \alpha_{k+1}}\right) +$$
$$+ \left(a^{(+1)} \left(\overset{k}{v}_x\right)^2, v^{\alpha_n - \alpha_{k+1}}\right)^+,$$

we obtain the basic integral (energy) identity

$$(\bar\rho, \overset{n}{v})^*_{\bar t} + 2I_n + P_n + 2^n (d, \overset{n}{v})^* = 2^n (Q_1(v), v^{\alpha_n})^* + \tag{40}$$
$$+ 2^n (\bar\psi, v^{\alpha_n})^* + (\bar\rho_{\bar t}, \overset{\vee}{\overset{n}{v}})^*,$$

where

$$I_n = \left(a, \left(\overset{n-1}{v}_{\bar x}\right)^2\right] + \sum_{k=0}^{n-2} 2^{n-k-2} \left\{\left(a \left(\overset{k}{v}_{\bar x}\right)^2, v^{\alpha_n - \alpha_{k+1}}\right) + \left(a^{(+1)} \left(\overset{k}{v}_x\right)^2, v^{\alpha_n - \alpha_{k+1}}\right)^+\right\}, \tag{41}$$

$$P_n = \tau \sum_{k=0}^{n-1} 2^{n-k-1} \left(\bar\rho \left(\overset{k}{v}_{\bar t}\right)^2, v^{\alpha_n - \alpha_{k+1}}\right)^*. \tag{42}$$

Our aim is to find an estimate for $\| v \|_0$ (and, therefore, for $\| z \|_0$) in terms of $\| \psi \|_3$. In [5] for the special case $b_{sk} = g_{sk} = 0$ for $s \neq k$ and a uniform net we obtained the estimate

$$\| \overset{n}{z}(x, t) \|_1^{1/2^n} \leqslant M_2 2^n e^{M_1 \varkappa 2^n} \| \overline{\psi(x, t)} \|_3, \quad \| \overline{\psi(x, t)} \|_3 = \max_{0 < t' \leqslant t} \| \psi(x, t') \|_3,$$

where $M_1 = 0$ for $g_{sk} = 0$, $s, k = 1, 2$. We shall show below that $M_1 = 0$ for the general equation (39).

We first find an *a priori* estimate in the mean, i.e. for $\| v \|_{2^n}$, and then use it to estimate $\| \overset{n}{v} \|_1$ and $\| v \|_0$.

7. *Stability in the mean with respect to the right-hand side.* Putting $n = 1$ in (40) we obtain the first rank energy identity:

$$(\bar\rho, v^2)^*_{\bar t} + 2I_1 + P_1 + 2(d, v^2)^* = 2(Q_1(v), v)^* + 2(\bar\psi, v)^* + (\bar\rho_{\bar t}, \check{v}^2)^*. \tag{43}$$

Repeating the argument made in Point 5, Section 1 of [5] for $n = 1$ we arrive, in the case of a non-uniform net, at the inequality

$$(\bar{\rho},\ v^2)^{*j+1} + \sum_{j'=1}^{j+1} \tau_{j'} (a,\ v_{\bar{x}}^2]^{j'} \leqslant M \| \overline{\psi (x,\ t_{j+1}) \|_3} \text{ for } \| \tau \|_0 < \tau_0.$$

It follows that

$$\| v (x,\ t) \|_{2^*} + \sqrt{\tau} \| v_{\bar{x}} (x,\ t) \|_2 \leqslant M \overline{\| \psi (x,\ t) \|_3} \text{ for } \| \tau \|_0 \leqslant \tau_0, \tag{44}$$

$$\| z (x,\ t) \|_{2^*} + \sqrt{\tau} \| z_{\bar{x}} (x,\ t) \|_2 \leqslant M \overline{\| \psi (x,\ t) \|_3} \text{ for } \| \tau \|_0 \leqslant \tau_0, \tag{45}$$

where M is a positive constant depending only on $c_1,\ c_2,\ \ldots,\ c_6,\ T$ and τ_0 is a positive constant depending only on $c_1,\ \ldots,\ c_5$.

Thus the scheme (39) is stable in the mean with respect to its right-hand side for sufficiently small $\tau \leqslant \tau_0$.

Lemma 5. If $d_1 \leqslant 0,\ b_{ss} = g_{ss} = 0,\ s = 1,\ 2,$ i.e.

$$Q (z) = Q^*(z) + d_1 z, \qquad Q^* (z) = b_{12} \check{z}_{\hat{x}} + b_{21} \check{z}_{\check{x}} + (g_{12} \check{z})_{\hat{x}} + (g_{21} \check{z})_{\check{x}} + d_2 \check{z}, \tag{46}$$

then the scheme (39) is stable in the mean with respect to its right-hand side on an arbitrary net Ω^* for any values of $\| h \|_0$ and $\| \tau \|_0$:

$$\| z (x,\ t) \|_{2^*} + \sqrt{\tau} \| z_{\bar{x}} (x,\ t) \|_2 \leqslant M \overline{\| \psi (x,\ t) \|_3}. \tag{47}$$

We start from identity (43), which we rewrite in the form

$$\begin{gathered}(\rho,\ v^2)_{\bar{t}}^* + \tau (\rho,\ v_{\bar{t}}^2)^* + 2I_1' + 2\overline{M} \| (\rho,\ v^2)^* \|_{2^*}^2 - 2 (1 + \overline{M}\tau) (d_1,\ v^2)^* = \\ = 2 (Q^* (v),\ v)^* + 2 (\psi',\ v) + (\rho_{\bar{t}},\ \check{v}^2)^*, \\ I_1' = (a',\ v_{\bar{x}}^2],\quad a' = a(1 + \overline{M}\tau),\quad \psi' = \overline{\psi}(1 + \overline{M}\tau).\end{gathered} \tag{48}$$

Let us estimate the separate terms of the right-hand side of (48):

$$2 (\psi',\ v)^* \leqslant 2 \| \psi' \|_3 \| v_{\bar{x}} \|_2 \leqslant \tfrac{1}{2} \check{I}_1' + \frac{2}{\bar{c}_1} \| \psi' \|_3^2, \qquad \bar{c}_1 = c_1 (1 + \overline{M}\tau) > c_1,$$

$$2 (d_2 \check{v},\ v)^* \leqslant 2c_3 \| \check{v} \|_{2^*} \| v \|_{2^*} \leqslant \frac{c_3}{\sqrt{c_2}} (\| \sqrt{\check{\rho}}\, \check{v} \|_{2^*}^2 + \| v \|_{2^*}^2),$$

$$2 (b_{12} \check{v}_{\hat{x}} + b_{21} \check{v}_{\check{x}},\ v)^* = 2 (b_{12}\ \check{v}_x,\ v)^+ + 2 (b_{21},\ \check{v}_{\bar{x}},\ v) \leqslant 4 \sqrt{2}\, c_4 \| \check{v}_{\bar{x}} \|_2 \| v \|_{2^*} \leqslant$$

$$\leqslant \frac{4\sqrt{2}c_4}{\sqrt{c_1}}\|v\|_{2^*}(\check{I}')^{1/2} \leqslant \frac{\check{\tau}}{\tau}\check{I}'_1 + \frac{16c_4^2}{c_1}\left(\frac{\tau}{\check{\tau}}\right)^2\|v\|^2_{2^*},$$

$$2\left((g_{12}\check{v})_{\hat{x}_i}, v\right)^* = -2\left(g_{12}\check{v}, v_{\bar{x}}\right] \leqslant \frac{2\sqrt{2}}{\sqrt{c_1}}c_5\|\check{v}\|_{2^*}(I'_1)^{1/2} \leqslant \frac{1}{4}I'_1 +$$
$$+\frac{4c_5^2}{c_1c_2}\|\sqrt{\check{\rho}}\check{v}\|^2_{2^*},\ (\rho_{\bar{t}}, \check{v}^2)^* \leqslant \frac{c_6}{c_2}\|\sqrt{\check{\rho}}, \check{v}\|^2_{2^*}.$$

Then choosing $\overline{M} = 0.5\,(c_3c_5^{-1/2} + 16c_4^2(m^*)^2/c_1)\,c_2^{-1}$, we arrive at the inequality

$$\|\sqrt{\rho}v\|^2_{2^*} + \tau I'_1 \leqslant (1 + M\tau)\|\sqrt{\check{\rho}}\check{v}\|^2_{2^*} + \check{\tau}\check{I}'_1 + \frac{2\tau}{c_1}\|\psi'\|^2_3,$$

which is valid for any τ on the net Ω^*.

Using Lemma 4a we find

$$\sqrt{\rho}v(x, t_{j+1})\|^2_{2^*} + \tau I'_1(t_{j+1}) \leqslant M\Big\{\|\sqrt{a'}v_{\bar{x}}\|^2_{2^*} + \|\sqrt{\rho^t}v\|^2_{2^*}|_{t=0} +$$
$$+\frac{2}{c_1}\sum_{j'=1}^{j+1}\tau_{j'}\|\psi'(x, t_{j'})\|^2_3\Big\}.$$

It follows that $\|v\|_{2^*} + \sqrt{\tau}\|v_{\bar{x}}\|_2 \leqslant M\|\overline{\overline{\psi}}\|_3$, since $v(x, 0) = 0$ and, using (38), we have inequality (47).

8. *Uniform stability*. Let us now derive an *a priori* estimate for $\|v\|_0$ from (40) using the method of majorant estimate of the right-hand side of the identity (40) suggested in [1] and perfected in [5]. The main difference between our method and that of [5] consists in our use of the estimate (44).

We need the formula

$$(v^{\alpha_n})_x = \sum_{k=0}^{n-1}(v^{(+1)})^{\alpha_k}v^{\alpha_n-\alpha_{k+1}}\overset{k}{v_x}$$

and a similar formula for $(v^{\alpha_n})_{\bar{x}}$. Consider the expression $2^n(\bar{\psi}, v^{\alpha_n})^*$. Putting $\bar{\psi} = \eta_{\check{x}}$, $\eta_N = 0$, we find

$$2^n(\bar{\psi}, v^{\alpha_n})^* = -2^n(\eta, (v^{\alpha_n})_x)^+$$

and, by analogy with [5]

$$2^n |((v^{\alpha_n})_x, \eta)^+| \leqslant \tfrac{1}{8} I_n + (M \times 2^n \|\eta\|_2)^{2^n} = \tfrac{1}{8} I_n + (M \times 2^n \|\bar{\psi}\|_3)^{2^n}. \quad (49)$$

This inequality allows us to write down an estimate for $2^n((g_{22}v)_{\check{x}}, v^{\alpha_n})^*$ immediately. For we have

$$2^n((g_{22}v)_{\check{x}}, v^{\alpha_n})^* = -2^n(g_{22}v, (v^{\alpha_n})_x)^+ \leqslant \tfrac{1}{8} I_n + (M \times 2^n \|v\|_{2^*})^{2^n}$$

and similarly for the other expressions of the same type, so that

$$K_n = 2^n((g_{11}v)_{\hat{x}} + (\bar{g}_{12}\check{v})_{\hat{x}} + (g_{22}v)_{\check{x}} + (\bar{g}_{21}\check{v})_{\check{x}}, v^{\alpha_n})^* \leqslant \\ \leqslant \tfrac{1}{8} I_n + [M 2^n (\|v\|_{2^*} + \|\check{v}\|_{2^*})]^{2^n}.$$

Since, from (44), $\|v\|_{2^*} \leqslant M \overline{\|\bar{\psi}\|_3}$,

$$K_n \leqslant \tfrac{1}{8} I_n + (M \times 2^n \overline{\|\bar{\psi}\|_3})^{2^n}, \qquad M = M(c_1, \ldots, c_5) > 0. \quad (50)$$

By analogy with [5] we find

$$2^n(b_{11}v_{\hat{x}} + b_{22}v_{\check{x}}, v^{\alpha_n})^* \leqslant \tfrac{1}{8} I_n + M \times 2^n (1, \overset{n}{v})^*, \qquad M = M(c_1, c_4) > 0, \quad (51)$$

$$2^n(\bar{d}_2\check{v}, v^{\alpha_n})^* \leqslant (\check{\bar{\rho}}, \overset{\check{n}}{v})^* + M \times 2^n (1, \overset{n}{v})^*, \qquad M = M(c_2, c_3) > 0. \quad (52)$$

Let us consider the estimate of the expression $2^n(\bar{b}_{12}\check{v}_{\hat{x}} + \bar{b}_{21}\check{v}_{\check{x}}, v^{\alpha_n})^*$ in more detail. Let us give the calculations for the first term only:

$$2^n(\bar{b}_{12}\check{v}_{\hat{x}}, v^{\alpha_n})^* = 2^n(\bar{b}_{12}\check{v}_x, v^{\alpha_n})^+ = 2^n\left(\bar{b}_{12}\check{v}_x, \left(\overset{\check{n-1}}{v} + \tau \overset{n-1}{v}_{\bar{t}}\right), v^{\alpha_{n-1}}\right)^+ = \\ = 2^n(\bar{b}_{12}\check{v}_x \check{v}^{\alpha_{n-1}}\check{v}, v^{\alpha_{n-1}})^+ + 2^n\tau(\bar{b}_{12}\check{v}_x, \overset{n-1}{v}_{\bar{t}} v^{\alpha_{n-1}})^+.$$

We see from formula (41) for I_n that

$$(\sqrt{a^{(+1)}}|\check{v}_x|, |v^{\alpha_{n-1}}|)^+ \leqslant (a^{(+1)}\check{v}_x^2, v^{\alpha_n - \alpha_1})^{1/2} \leqslant 2^{-n/2+1} I_n^{1/2}.$$

We can therefore write

$$2^n(\bar{b}_{12}\check{v}_x\overset{n-1}{\check{v}}, v^{\alpha_{n-1}})^+ \leqslant 2^n c_4 (\check{v}_x^2, \overset{\check{n}}{v})^{+1/2} (v^{\alpha_{n-1}}, 1)^{+1/2} \leqslant \\ \leqslant 2^n c_4 c_1^{-1/2^n} \check{I}_n^{1/2^n} (\check{v}_x^2, \check{v}^{\alpha_n - \alpha_1})^{+1/2} (1, \overset{n}{v})^{+(1/2 - 1/2^n)} \leqslant \quad (53)$$

$$\leqslant 2^{(n+1)/2}c_4c^{-(1/2-1/2^n)}\check{I}_n^{1/2+1/2^n}(1,\overset{n}{v})^{+(1/2-1/2^n)}\leqslant$$

$$\leqslant \frac{1}{2}\frac{\check{\tau}}{\tau}\check{I}_n+M2^n(1,\overset{n}{v})^*,\qquad M=M(c_1,\ c_4,\ m^*)>0,$$

since $\overset{n}{v}=v^{\alpha_{n-1}}v^2$, $v^2\leqslant\left(\frac{1}{c_1}I_n\right)^{1/2^{n-1}}$, $\left(\check{v}_x^2,\ \check{v}^{\alpha_n-\alpha_1}\right)^+\leqslant 2^{(n-1)/2}\left(\check{I}_n/c_1\right)^{1/2}$.

Now let us consider the second expression:

$$2^n\tau(\bar{b}_{12}\check{v}_x,\ \overset{n-1}{v_{\bar t}}v^{\alpha_{n-1}})^+\leqslant\frac{2^nc_4\sqrt{2}}{\sqrt{c_2}}\left(\frac{1}{c_1}I_n\right)^{1/2-1/2^n}(\sqrt{\tau}\|\check{v}_{\bar x}\|_2)(\sqrt{\tau}\|\sqrt{\bar\rho}\,\overset{n-1}{v_{\bar t}}\|_{2^*})\leqslant$$

$$\leqslant\frac{1}{8}I_n+\frac{1}{2}\tau\left\|\sqrt{\bar\rho}\,\overset{n-1}{v_{\bar t}}\right\|_{2^*}^2+(M\times 2^n\sqrt{\tau}\|\check{v}_{\bar x}\|_2)^{2^n}.$$

Then, using estimate (44) for $\sqrt{\tau}\|\check{v}_{\bar x}\|_2$, we find

$$2^n(\bar{b}_{12}\check{v}_{\hat x}+\bar{b}_{21}\check{v}_{\check x},\ v^{\alpha_n})^*\leqslant\frac{1}{4}I_n+P_n+\frac{\check\tau}{\tau}\check{I}_n+(M\times 2^n\overline{\|\bar\psi\|_3})^{2^n}.$$

Combining all the estimates (51)-(53) and choosing $\bar{M}$ so that the coefficient of $\left(1,\overset{n}{v}\right)^*$ in (40) is non-negative, we obtain the inequality

$$(\bar\rho,\ \overset{n}{v})^*+\tau I_n\leqslant(\check{\bar\rho},\ \overset{n}{\check v})^*(1+M_1\tau)+\check\tau\check{I}_n+\tau(M\times 2^n\overline{\|\bar\psi\|_3})^{2^n},\tag{54}$$

which is valid for sufficiently small $\tau<\tau_0$, where τ_0, M_1, M are positive constants independent of the net, $\tau_0=\tau_0(c_1,\ \ldots,\ c_5)$, $M_1=M(c_2,\ c_6)$, $M=M(c_1,\ \ldots,\ c_5,\ m^*)$.

Using Lemma 4a, it follows from (54) that

$$(1,\ \overset{n}{v})^*+\tau\left\|\overset{n-1}{v_{\bar x}}\right\|_2^2\leqslant M'(M2^n\overline{\|\bar\psi\|_3})^{2^n}\quad\text{for }\tau\leqslant\tau_0.\tag{55}$$

Going from v to z:

$$(1,\ \overset{n}{z})^*+\tau\left\|\overset{n-1}{z_{\bar x}}\right\|_2^2\leqslant M''(M2^n\overline{\|\bar\psi\|_3})^{2^n}\quad\text{for }\tau\leqslant\tau_0.$$

We then obtain

$$\|z(x,\ t)\|_0\leqslant M\times 2^n\overline{\|\psi(x,\ t)\|_3}\hbar_*^{-1/2^n},\quad \hbar_*=\min_{0<i<N}\hbar_i,\tag{56}$$

$$\|z(x,\ t)\|_0\leqslant M\times 2^n\overline{\|\psi(x,\ t)\|_3}\tau_*^{-1/2^n},\quad \tau_*=\min_{0<t_j\leqslant t}\tau_j.\tag{57}$$

The second inequality follows from

$$\left(\tau \left\| \overset{n-1}{z_{\bar{x}}} \right\|_2^2\right)^{1/2^n} \leqslant M \times 2^n \overline{\| \psi(x, t) \|_3}, \qquad \left\| \overset{n}{z} \right\|_0 \leqslant \left\| \overset{n-1}{z_{\bar{x}}} \right\|_2^2.$$

Choosing n so that condition (78), Section 1 of [5], i.e. the condition

$$\frac{\log_2 \frac{1}{\hbar_*}}{\varepsilon \log_2 \log_2 \frac{1}{\hbar_*}} \leqslant 2^n \leqslant \log_2 \frac{1}{\hbar_*} \text{ for } \hbar_* \leqslant h_0(\varepsilon), \tag{58}$$

is satisfied, where $\varepsilon > 0$ is an arbitrary number, and using the relation

$$n + \frac{1}{2^n} \log_2 \frac{1}{\hbar_*} \leqslant \delta \log_2 \log_2 \frac{1}{\hbar_*}, \quad \delta = 1 + \varepsilon > 1,$$

we find from (56)

$$\| z(x, t) \|_0 \leqslant M \overline{\| \psi(x, t) \|_3} \ln^\delta \frac{1}{\hbar_*} \quad \text{for } \hbar_* \leqslant h_0(\varepsilon). \tag{59}$$

Choosing $n = n(\tau_*)$ by analogy with (58) and using (57) we obtain

$$\| z(x, t) \|_0 \leqslant M \overline{\| \psi(x, t) \|_3} \ln^\delta \frac{1}{\tau_*} \quad \text{for } \tau_* \leqslant \tau_0(\varepsilon).$$

This proves the following theorem.

Theorem 5. If $z = z(x, t)$ is the solution of problem (33) and conditions (34) and (35) are satisfied, then on the arbitrary sequence of nets Ω we have the following estimates:

$$\| z(x, t) \|_2 \leqslant M \overline{\| \psi(x, t)_3 \|} \qquad \text{for } \| \tau \|_0 \leqslant \tau_0 \text{ (estimation on average)} \tag{60*}$$

$$\| z(x, t) \|_0 \leqslant M \overline{\| \psi(x, t) \|_3} \ln^\delta \frac{1}{\tau_*} \quad \text{for } \| \tau \|_0 \leqslant \tau_0'(\tau_0, \varepsilon), \tag{61}$$

$$\| z(x, t) \|_0 \leqslant M \overline{\| \psi(x, t) \|_3} \ln^\delta \frac{1}{\hbar_*} \quad \text{for } \| \tau \|_0 \leqslant \tau_0, \ \hbar_* \leqslant h_0(\varepsilon), \tag{62}$$

where $\delta = 1 + \varepsilon$, ε is an arbitrary positive number, and τ_0, M are positive constants which do not depend on the net.

Note 1. If $b_{12} = b_{21} = 0$ then Theorem 5 is true for an arbitrary net Ω.

Note 2. If $b_{ss} = g_{sk} = 0$ $(k, s = 1, 2)$ then the principle of the maximum gives

$$\| z(x, t) \|_0 \leqslant M \overline{\| \psi(x, t) \|_0} \text{ for } \| \tau \|_0 \leqslant \tau_0$$

(cf. [5]) on any net Ω. If, in addition, $d_1 \leqslant 0$, this estimate is valid for all $\|\tau\|_0$.

4. On the order of accuracy of difference schemes on non-uniform nets

1. *Introduction.* In Section 2 we studied various homogeneous difference schemes for linear and quasi-linear parabolic equations in the case of non-uniform nets. Our final aim is to find the order of accuracy of these schemes on arbitrary non-uniform nets in the class of continuous and discontinuous coefficients of the corresponding differential equations. This problem is studied in detail in [2], [4], [5] for uniform nets. The *a priori* estimates obtained in Section 3 enable us to strengthen Theorem 6 of [5] concerning the order of accuracy of schemes for the quasi-linear equation (2.21) considerably, even in the case of uniform nets. The two-parameter family of difference schemes (2.22) obviously includes the schemes studied in [5].

We shall always assume that in all the region $\overline{\mathcal{D}}$ or in each of the regions $\bar{\Delta}_\nu$, $\nu = 0, 1, \ldots, \nu_0$ (in the case of discontinuous coefficients) the conditions under which the schemes we are studying have the maximum order of approximation on uniform nets are satisfied. Then, as we showed in Section 2, the approximation error Ψ of our schemes can be put in the form

a) for the schemes (2.10), (2.17) and (2.19)

$$\Psi = \mu^{(\alpha)}_{\hat{x}} + \psi^*, \quad \psi^* = O(h^2) + O(h_+^2) + O(\tau^{m_\alpha}), \quad m_\alpha = \begin{cases} 2, & \alpha = 0.5 \\ 1, & \alpha \neq 0.5 \end{cases} \qquad \mu = \mu_0 h^2 = O(h^2), \tag{1}$$

where μ_0 is given by one of the formulae (2.10), (2.18);

b) for the scheme (2.22)

$$\Psi = \psi = \mu_{\hat{x}} + \psi^*, \quad \mu = \mu_0 h^2 = O(h^2), \quad \psi^* = O(h^2) + O(h_+^2) + O(\tau). \tag{2}$$

If conditions A are satisfied in $\overline{\mathcal{D}}$, then (1) and (2) are valid at all nodes (x, t) of the net Ω. If there are lines of discontinuity of the differential equation $x = \xi_\nu(t) = x_{n_\nu} + \theta_\nu h_{n_\nu+1}$ $(0 \leqslant \theta_\nu \leqslant 1$, $\nu = 1, 2, \ldots, \nu_0)$ and conditions A are satisfied in each region $\bar{\Delta}_\nu$, then in the case of fixed discontinuities $(\xi'_\nu(t) \equiv 0$, $\nu = 1, 2, \ldots, \nu_0)$ (1) and (2) are valid at all nodes of the net Ω apart from the nodes (x_{n_ν}, t_j)

and $(x_{n_\nu+1}, t_j)$, $j = 0, 1, \ldots, K$. We can choose the sequence of nets $\Omega(k)$ in such a way that the formulae (1) and (2) hold along the lines of discontinuity $x = \xi_\nu = x_{n_\nu}$ also, for these nets.

In the case of moving or oblique discontinuities ($\xi'_\nu(t) \neq 0$ for at least one ν) the situation is more complicated (cf. [2]).

We shall assume below that in the case (a) the following condition is always satisfied: the function $\mu_0(x, t)$ satisfies the Lipschitz condition in t, so that

$$|(\mu_0)_{\bar{t}}| \leqslant M, \qquad \mu_{\bar{t}} = O(h^2), \tag{3}$$

where M is a positive constant which does not depend on the net.

It is clear from formula (2) that (3) implies, for instance, that $\partial^2 u/\partial x^3$ satisfies the Lipschitz condition in t.

To simplify our formulations, instead of saying "the solution of problem (2.7) converges uniformly to the solution of problem (2.1) and has an accuracy of $O(h^2) + O(\tau^{m\alpha})$" we shall say: "the scheme (2.7) converges uniformly at a rate $O(h^2) + O(\tau^{m\alpha})$".

This enables us to simplify the formulations of the theorems concerning the convergence and accuracy of our schemes.

2. *Continuous coefficients.*

Theorem 6. When $0.5 \leqslant \alpha \leqslant 1$ the difference scheme (2.17) converges uniformly at a rate $O(\|h^2\|_2 + O(\|\tau^{m\alpha}\|_2)$ on an arbitrary sequence of non-uniform nets $\bar{\Omega}$, or, more accurately

$$\|y - u\|_0 \leqslant M(\|h^2\|_2 + \|\tau^{m\alpha}\|_2) \quad \text{for } \|\tau\|_0 \leqslant \tau_0, \tag{4}$$
$$\|\tau^{m\alpha}\|_2 = \left(\frac{1}{t}\sum_{t_{j'}=t_1}^{t} (\tau_{j'}^{m\alpha})^2 \tau_{j'}\right)^{1/2}, \qquad \|h^2\|_2 = (1, h^2]^{1/2},$$

where y is the solution of problem (2.17), u is the solution of problem (2.13), τ_0 and M are positive constants which do not depend on the choice of nets.

Theorem 7. When $0 \leqslant \alpha \leqslant 1$ and $0 \leqslant \beta \leqslant 1$ the difference schemes (2.2) converge in the mean at a rate $O(\|h^2\|_2) + O(\|\tau\|_2)$ and converge uniformly on the sequence of nets Ω^* at a rate $O\left(\|h^2\|_2 \ln^5 \frac{1}{h_*}\right) + O\left(\|\tau\|_2 \ln^5 \frac{1}{\tau_*}\right)$, or, more accurately

$$\| y - u \|_2 = \| y(x,t) - u(x,t) \|_2 \leqslant M (\| h^2 \|_2 + \| \tau \|_2) \quad \text{for } \| \tau \|_0 \leqslant \tau_0, t \in \omega_\tau, \tag{5}$$

$$\| y(x,t) - u(x,t) \|_0 \leqslant M \left(\| h^2 \|_2 \ln^\delta \frac{1}{\hbar_*} + \| \tau \|_2 \ln^\delta \frac{1}{\tau_*} \right)$$
$$\text{for } \| \tau \|_0 \leqslant \tau_0'(\tau_0, \delta), \quad \hbar_* \leqslant h_0(\delta), \quad \tau \leqslant m^* \check{\tau}, \tag{6}$$

where u is the solution of problem (2.21), y is the solution of problem (2.22), $\hbar_* = \min\limits_{0<i<N} \hbar_i$, $\tau_* = \tau_{*j+1} = \min\limits_{0<j'<j+1} \tau_{j'}$, $\delta > 1$ is an arbitrary number and τ_0, M, m^* are constants which are independent of the choice of net $(t = t_{j+1})$.

To prove Theorem 6 it is sufficient to use expression (1) for Ψ and Theorem 4; to prove Theorem 7 we use expression (2) and Theorem 5 (cf. [2]-[5]).

If the net $\bar{\Omega}$ is uniform, then from (6) (cf. [5]) we have

$$\| y - u \|_0 \leqslant M \left(h^2 \ln^\delta \frac{1}{h} + \tau \ln^\delta \frac{1}{\tau} \right). \tag{7}$$

We are naturally assuming that as $\| h \|_0 \to 0$ and $\| \tau \|_0 \to 0$, $h_* = \min \hbar_i$ and $\tau_* = \min \tau_j$ satisfy the requirements: $\ln(1/\|h_*\|) < 1/\| h \|_0^\varepsilon$, $\ln(1/\tau) < 1/\| \tau \|_0^\varepsilon$, where ε is a positive number as small as we please.

3. *Discontinuous coefficients*. Let us restrict ourselves to studying the convergence of our schemes for fixed discontinuities only, when the coefficients of the differential equations have discontinuities of the first kind only on a finite number of straight lines parallel to the t-axis in the (x, t) plane.

Theorem 8. The difference scheme (2.17) converges uniformly on the sequence of nets $\bar{\Omega}(k)$ at a rate $O(\| h^2 \|_2) + O(\| \tau^{m_\alpha} \|_2)$ when the coefficients of equation (2.13) have only fixed discontinuities and $0.5 \leqslant \alpha \leqslant 1$.

If we choose the net $\bar{\Omega}(k) = \bar{\omega}_h(k) \times \bar{\omega}_\tau$ so that the lines of discontinuity of the coefficients of the differential equation (2.13) are nodal lines of the net $\bar{\Omega}(k)$, then, as we showed in Section 2, the approximation errors are given by (2.18) at all the nodes of the net $x \in \omega_h(k)$ and for all $t_j \in \omega_\tau$. Therefore the proof of Theorem 8 is the same as that of Theorem 6.

We shall not formulate the theorems about the accuracy of the schemes (2.17) and (2.22) on an arbitrary sequence of nets $\bar{\Omega}$ (or $\bar{\Omega}^*$) since in this case the corresponding theorems of [5] apply, if we put

$O(h^{\varkappa}) + O(\tau^{m_\alpha})$ instead of $O(\|h\|_0^{\varkappa}) + O(\|\tau^{m_\alpha}\|_2)$, and for the scheme (2.22) put

$$O\left(\|h\|_0^{\varkappa}\ln^{\delta}\frac{1}{h_*}\right) + O\left(\|\tau\|_2\ln^{\delta}\frac{1}{\tau_*}\right) \tag{8}$$

instead of $O(h^{\varkappa_1-\rho(h)}) + O(\tau^{1-\rho(\tau)})$, where $\rho(h) \sim 1/\sqrt{\ln(1/h)}$, $\rho(\tau) \sim 1/\sqrt{\ln(1/\tau)}$.

The generalisation of the theorem of [5] to the case of non-uniform nets also presents no difficulty. From Theorem 5 we know that the schemes (2.7), (2.17) and (2.22) have the same order of accuracy if $k(x, t)$ and, correspondingly, $k(x, t, u)$ possess moving (oblique) discontinuities (see, also, [2]).

4. *Other schemes for quasi-linear equations.* Consider problem (2.21). Besides the schemes mentioned in Section 2 the scheme

$$\left.\begin{aligned} \rho y_{\bar t} &= \tfrac{1}{2}\left(a_{(0.5)}(x, t, y^*)\left(y_{\bar x} + \check y_{\bar x}\right)\right)_{\hat x} + \varphi\left(x, \bar t, \tfrac{1}{2}(y + \check y), \tfrac{1}{2}\lambda(y + \check y)\right), \\ y_0 &= u_1(t), \qquad y_N = u_2(t), \qquad y(x, 0) = u_0(x), \end{aligned}\right\} \tag{9}$$

where $\rho = \rho(x, \bar t)$, $\bar t = t - 0.5\,\tau$, and $a_{(0.5)}$ is found from the second of the formulae of Point 5, Section 2 with $\alpha = 0.5$, $y^* = 0.5(y + y^{(-1)})$, deserves attention.

On a uniform net this scheme has second order approximation. On a non-uniform net the approximation error can be put in the form

$$\Psi = \mu_{\hat x} + \psi^*, \quad \mu = O(h^2) + O(\tau^2), \quad \psi^* = O(h^2) + O(h_+^2) + O(\tau^2). \tag{10}$$

The error $z = y - u$, where y is the solution of problem (9), is given by a special case (with $b_1 = b_2$, $g_1^{(-1)} = g_2$) of the problem

$$\left.\begin{aligned} &\rho z_{\bar t} = 0.5\Lambda(z + \check z) + \Psi, \qquad z_0 = z_N = 0, \qquad z(x, 0) = 0, \\ &\Lambda w = (a w_{\bar x})_{\hat x} + b_1 w_{\hat x} + b_2 w_{\check x} + (g_1 w)_{\hat x} + (g_2 w)_{\check x} + dw, \end{aligned}\right\} \tag{11}$$

$$0 < c_1 \leqslant a, \quad 0 < c_2 \leqslant \rho, \quad |d| \leqslant c_3, \quad |b_s| \leqslant c_4, \quad |g_s| \leqslant c_5, \quad s = 1, 2, \quad |\rho_{\bar t}| \leqslant c_6. \tag{12}$$

Let us show that the scheme (11)-(12) is stable in the mean on any sequence of nets, so that for $\|\tau\|_0 \leqslant \tau_0$ we have the estimate

$$\|z(x, t_{j+1})\|_{2*} \leqslant M\left\{\|\sqrt{\rho(x, 0)}\, z(x, 0)\|_{2*} + \left[\sum_{j'=1}^{j+1}\tau_{j'}\|\Psi(x, t_{j'})\|_3^2\right]^{1/2}\right\}. \tag{13}$$

As usual, we go from the function z to the function v putting

$$z^j = v^j\mu^j, \qquad \mu = (1 + \overline{M}\tau)\check{\mu} \quad \text{or} \quad \mu_{\bar{t}} = \overline{M}\check{\mu}, \ \mu(x, 0) = 1 \qquad (14)$$

where $\overline{M}$ is an arbitrary positive constant.

To simplify the calculations we shall take $\rho = 1$ for the time being. Using the relation $z_{\bar{t}} = 0.5\,(\mu + \check{\mu})\,v_{\bar{t}} + 0.5\,(v + \check{v})\,\mu_{\bar{t}}$, and the notation

$$w = \frac{\mu v + \check{\mu}\check{v}}{\mu + \check{\mu}},$$

and noting that $v + \check{v} = 2w - \varkappa\tau^2 v_{\bar{t}}$, we obtain

$$(1 - \tau^2\varkappa^2)v_{\bar{t}} + 2\varkappa w = \Lambda w + \overline{\Psi}, \qquad (15)$$

where

$$\varkappa = \mu_{\bar{t}}/(\mu + \check{\mu}), \qquad \overline{\Psi} = \Psi/0.5\,(\mu + \check{\mu}).$$

Consider the product

$$v_{\bar{t}}w = v_{\bar{t}}(\mu v + \mu v)/(\mu + \check{\mu}) = 0.5(\mu + \check{\mu})^{-1}\{\mu[(v^2)_{\bar{t}} + \tau v_{\bar{t}}^2] + \check{\mu}[(v^2)_{\bar{t}} - \tau v_{\bar{t}}^2]\} = 0.5(v^2)_{\bar{t}} + 0.5\tau^2\varkappa v_{\bar{t}}^2. \qquad (16)$$

It is easy to see that

$$1 - \tau^2\varkappa^2 = 1 - \tau^2(\mu_{\bar{t}})^2(\mu + \check{\mu})^{-2} = 1 - (\mu - \check{\mu})^2\,(\mu + \check{\mu})^{-2} = 2\mu\check{\mu}(\mu + \check{\mu})^{-2} > 0. \qquad (17)$$

We multiply (15) by wh and sum with respect to $x \in \omega_h$. Using (16), (17), after a series of simple majorant estimates, we arrive at the inequality

$$(\rho, v^2)^* \leqslant (1 + M\tau)\,(\check{\rho}, \check{v}^2)^* + \tau\|\Psi\|_3^2(c_1\mu\check{\mu})^{-1} \qquad (18)$$

(we have written it for $\rho \neq 1$), and this will be satisfied if we choose $\overline{M} = \overline{M}(c_1, \ldots, c_5)$ sufficiently large and τ sufficiently small:

$$\tau \leqslant \tau_0, \qquad \tau_0 = \tau_0\,(c_1, \ldots, c_5) > 0.$$

Using Lemma 4a and returning to the function z we obtain (13).

We can use the estimate (13) to prove the theorems concerning the convergence and accuracy of the scheme (9) both in the class of continuous coefficients of equation (2.21) and in the class of discontinuous coefficients of this equation. We note only that in the class of continuous k and f (conditions A are satisfied in $\overline{Д}$) the scheme (9) has second order accuracy:

$$\| y - u \|_{2^*} = O(\| h^2 \|_2) + O(\| \tau^2 \|_2) \qquad \text{on } \overline{\Omega}.$$

The scheme (0) has the same order of accuracy on the sequence of nets $\overline{\Omega}(k)$ in the case of fixed discontinuities also (the conditions A are satisfied in each of the regions $\overline{\Delta}_\nu$, $\nu = 0, 1, \ldots, \nu_0$).

5. *On an economical homogeneous difference scheme for a system of parabolic equations.* Let us consider in $\overline{Д} = 0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant T)$ the problem

$$\left.\begin{aligned} &\frac{\partial u^i}{\partial t} = \sum_{j=1}^{p} \frac{\partial}{\partial x}\left(k_{ij}(x, t)\frac{\partial u^j}{\partial x}\right) + f^i(x, t) \qquad (i = 1, 2, \ldots, p),\\ &u^i(0, t) = u_1^i(t), \quad u^i(1, t) = u_2^i(t), \quad u^i(x, 0) = u_0^i(x), \quad i = 1, \ldots, p; \end{aligned}\right\} \tag{19}$$

$$\sum_{i,j=1}^{p} k_{ij}\xi_i\xi_j \geqslant c_1 \sum_{i=1}^{p} \xi_i^2, \quad k_{ij} = k_{ji}, \ |(k_{ij})_{\bar{t}}| \leqslant c_2 \ (c_m = \text{const} > 0, \ m = 1, 2). \tag{20}$$

In [3] we studied six-point schemes for a system of parabolic equations and obtained *a priori* estimates which, by analogy with the case of one equation ($p = 1$), enable us to prove uniform convergence and to obtain an estimate of the order of accuracy of these schemes both in the class of continuous coefficients and in the class of coefficients which have fixed discontinuities. There is no need to give the proofs here. Since the schemes of [3] are implicit, the solution of the resulting difference equations requires a larger amount of calculation and can be found, for example, by using the formulae of matrix successive substitution [9]. We consider below a scheme which requires a small number of operations, since in order to find the vector $y = \{y^i\}$ at each moment of time $t = t_{n+1}$ we require only a p-times successive application of the one-dimensional substitution formulae (cf. [9], [10], [4]). This scheme has the form

$$\left.\begin{aligned} y_{\bar{t}}^i = &\sum_{j=1}^{i-1} (a_{ij} y_{\bar{x}}^j)_{\hat{x}} + \tfrac{1}{2}[(\check{a}_{ii}\check{y}_{\bar{x}}^i)_{\hat{x}} + (a_{ii} y_{\bar{x}}^i)_{\hat{x}}] + \\ &\qquad + \sum_{j=i+1}^{p} (\check{a}_{ij}\check{y}_{\bar{x}}^j)_{\hat{x}} + \varphi^i(x, t - 0.5\tau),\\ y^i(0, t) &= u_1^i(t), \quad y^i(1, t) = u_2^i(t), \quad y^i(x, 0) = u_0^i(x) \quad (i = 1, 2, \ldots, p). \end{aligned}\right\} \tag{21}$$

for the equations (19).

We shall restrict ourselves here to the class of schemes for which

$$a_{ij} = A\,[k_{ij}(x + sh, t)], \qquad -1 \leqslant s \leqslant 0,$$

where $A\,[\mu(s)]$ is a linear non-decreasing functional for which

$$A\,[1] = 1, \qquad A\,[s] = -0.5$$

It follows from this and (20) that

$$\sum_{i,\,j=1}^{p} a_{ij}\xi_i\xi_j = A\Big[\sum_{i,\,j=1}^{p} k_{ij}(x_i + sh, t)\,\xi_i\xi_j\Big] \geqslant c_1 \sum_{i=1}^{p} \xi_i^2. \tag{22}$$

To compute φ^i we use the same functional $F\,[\bar{f}(s)]$, that we used in Sections 1, 2:

$$\varphi^i(x, t) = \frac{h}{\hbar} F\,[f^i(x + sh, t)\,\eta_0^-(s)] + \frac{h_+}{\hbar} F\,[f^i(x + sh_+, t)\,\eta_0^+(s)].$$

The approximation error $\{\Psi^i\} = \Psi$ of scheme (21) can be put in the form

$$\left.\begin{aligned} \Psi^i &= (\mu^i)_{\hat{x}} + \psi^{*i}, \qquad \mu^i = O(h^2), \\ \psi^{*i} &= O(h^2) + O(h_+^2) + O(\tau), \qquad i = 1, 2, \ldots, p. \end{aligned}\right\} \tag{23}$$

By analogy with [3], using the method of energy inequalities, we obtain the *a priori* estimate

$$\|y - u\|_0 \leqslant M\,\{\overline{\|\mu(x, t)\|_2} + \overline{\|\mu_{\bar{t}}(x, t)\|_2} + \overline{\|\psi^*(x, t)\|_3}\} \tag{24}$$

for sufficiently small $\|\tau\|_0 \leqslant \tau_0$. Here $y = \{y^i\}$, $u = \{u^i\}$, $\mu = \{\mu^i\}$, $\psi^* = \{\psi^{*i}\}$ are net vector-functions.

It follows from this and from (23) that the scheme (21) converges uniformly on an arbitrary sequence of non-uniform nets at a rate $O(\|h^2\|_2) + O(\|\tau\|_2)$.

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