HOMOGENEOUS DIFFERENCE SCHEMES OF A HIGH DEGREE OF ACCURACY ON NON-UNIFORM NETS*

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The homogeneous difference schemes corresponding to the boundary problems for the differential equation

$$L^{(p,q,f)} u = \frac{d}{dx}\left[\frac{1}{p(x)}\frac{du}{dx}\right] - q(x)u + f(x) = 0,\ 0 < x < 1, \qquad (1)$$
$$M_2 \geqslant p(x) \geqslant M_1 > 0, \qquad 0 \leqslant q(x) \leqslant M_3,$$

with piece-wise continuous coefficients ($p,\ q,\ f \in Q^{(0)}$) include an exact scheme [1], [2]. This scheme enables us to determine the net function which coincides with the exact solution of the boundary problem on an arbitrary non-uniform net

$$S_N(x_0 = 0,\ x_1, \ldots, x_i, \ldots, x_N = 1,\ h_i = x_i - x_{i-1}).$$

In this article we construct schemes of any (pre-set) order of accuracy on non-homogeneous nets.

All the schemes will be of the form

$$L_h^{(p,q,f)} y_i = \frac{1}{\hbar_i}\Delta\left(\frac{\nabla y_i}{h_i A_i^h}\right) - D_i^h y_i + \Phi_i^h,$$
$$\hbar_i = 0{,}5\,(h_i + h_{i+1}), \qquad \Delta y_i = \nabla y_{i+1} = y_{i+1} - y_i. \qquad (2)$$

The upper index h is the conventional notation for the dependence between the coefficients and the net.

In the case of a uniform net ($h_i = h = 1/N$, $i = 1, 2, \ldots, N$) the difference schemes (2) belong to the family of homogeneous three-point

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conservative schemes [2], [3]. This is also true when the concepts of homogeneity and conservativeness of schemes are generalized to non-uniform nets.

Schemes of an increased degree of accuracy prove to be especially useful in a number of cases which are important in practice, such as in the solution of equation (1) with piece-wise constant coefficients with a large number of discontinuities, or in the solution of systems of such equations, or in the solution of equations of heat conduction and diffusion of the form $\partial u/\partial t = L^{(p,q,f)}u$.

1. An exact scheme on a non-uniform net

1. *The construction of the exact scheme*

Consider the first boundary problem for the differential equation

$$L^{(p,q,f)}u = 0,\ 0 < x < 1,\ u(0) = \mu_1,\ u(1) = \mu_2. \tag{3}$$

Let $S_N = \{x_i\}$ be some difference net, obtained as a result of dividing up the segment $0 \leqslant x \leqslant 1$ into N parts by the points

$$x_0 = 0, \quad x_1, \ldots, x_i, \ldots, x_N = 1.$$

At each point x_i there are two steps (left and right) of the difference net, $h_i = x_i - x_{i-1}$ and $h_{i+1} = x_{i+1} - x_i$, which in general are not equal to one another. Let $\hbar_i = 0.5\,(h_i + h_{i+1})$ be the mean step of the net at the point x_i. The local non-uniformity of the net can be characterized by the ratio

$$\Delta_i = -\frac{h_{i+1} - h_i}{h_{i+1} + h_i} = -\frac{0{,}5\,(h_{i+1} - h_i)}{\hbar_i}.$$

If the difference net is uniform, then $\Delta_i = 0$.

We shall assume henceforth that on any sequence of nets S_N the condition

$$0 < C_1 \leqslant \frac{h_{i+1}}{h_i} \leqslant C_2, \tag{H}$$

is satisfied, where C_1 and C_2 are positive constants independent of the net.

We shall use the notation

$$h^* = \max_{0<i\leqslant N} h_i.$$

Let us show that the solution of equation (3) on any net S_N satisfies the relation

$$u\ (x_i) = P_i^h u\ (x_{i-1}) + Q_i^h u\ (x_{i+1}) + R_i^h, \tag{4}$$

where P_i^h, Q_i^h, R_i^h are coefficients expressed in terms of some functionals depending on the net S_N (or S_h) or, to be more precise, on the two parameters h_l and h_r.

The relation (4) is obviously possible, since the solution of a differential equation of the second order in the interval (x_{i-1}, x_{i+1}) is completely determined by the values of the required function at the ends of this interval.

In the article [1] the exact scheme of (4) was constructed on a uniform net.

Let us consider equation (3) in the interval (x_{i-1}, x_{i+1}). We introduce the local system of co-ordinates associated with the point $x = x_i$ by putting

$$x = x_i + \hbar_i\ (s - \Delta_i), \quad -1 \leqslant s \leqslant 1,$$

or

$$x = \overline{x_i} + \hbar_i\ s, \text{where } \overline{x_i} = x_i - \hbar_i \Delta_i.$$

The segment $[x_{i-1}, x_{i+1}]$ is mapped into the segment $-1 \leqslant s \leqslant 1$, the point $x = x_i$ corresponding to the point $s = \Delta_i$. Equation (3) takes the form

$$\mathcal{L}^* \overline{u} = \frac{d}{ds}\left[\frac{1}{\overline{p}\ (s)} \frac{d\overline{u}}{ds}\right] - \hbar^2 \overline{q}\ (s) = -\hbar^2 \overline{f}\ (s), \tag{5}$$

where

$$\overline{u}\ (s) = u\ (x_i + \hbar_i\ (s - \Delta_i)) \qquad \text{etc.} \tag{6}$$

We omit the suffix i for all the functions in (5), as well as the step h.

The general solution of equation (5) in the segment $[-1, 1]$ has the form

$$\overline{u}\ (s) = C v_1\ (s, \hbar) + D v_2\ (s, \hbar) + \hbar^2 v_3\ (s, \hbar),$$

where C and D are arbitrary constants, $v_1(s, h)$ and $v_2(s, h)$ are linearly independent solutions of the homogeneous equation $\mathcal{L}^* u = 0$ and $v_3(s, h)$ is some solution of the non-homogeneous equation (5).

We determine the functions v_1, v_2 and v_3 from the conditions

$$\left.\begin{aligned} v_1(-1, \hbar) = 0, \quad \frac{1}{\bar{p}(-1)}\frac{dv_1}{ds}(-1, \hbar) = 1, \\ v_2(1, \hbar) = 0, \quad \frac{1}{\bar{p}(1)}\frac{dv_2}{ds}(1, \hbar) = -1, \end{aligned}\right\} \tag{7}$$

$$v_3(-1, \hbar) = 0, \quad v_3(1, \hbar) = 0. \tag{8}$$

We shall call the functions v_1, v_2, v_3 *pattern functions*. They are functionals of the coefficients $\bar{p}(s)$, $\bar{q}(s)$ and $\bar{f}(s)$ and depend parametrically on h.

Writing $s = -1$ and $s = 1$ in (6) and using conditions (7) and (8) we find

$$C = \frac{\bar{u}(1)}{v_1(1, \hbar)}, \quad D = \frac{\bar{u}(-1)}{v_2(-1, \hbar)}$$

and therefore

$$\bar{u}(s) = \frac{v_2(s, \hbar)}{v_2(-1, \hbar)}\bar{u}(-1) + \frac{v_1(s, \hbar)}{v_1(1, \hbar)}\bar{u}(1) + \hbar^2 v_3(s, \hbar). \tag{9}$$

It is clear from this that the normalization of the functions v_1 and v_2 is arbitrary. Writing $s = \Delta$ in (9) we obtain the relation

$$\bar{u}(\Delta) = P^h\bar{u}(-1) + Q^h\bar{u}(1) + R^h, \tag{10}$$

the coefficients of which

$$P^h = \frac{v_2(\Delta, \hbar)}{v_2(-1, \hbar)} = P^{\hbar}[\bar{p}(s), \bar{q}(s), \Delta], \quad Q^h = \frac{v_1(\Delta, \hbar)}{v_1(1, \hbar)} = Q^{\hbar}[\bar{p}(s), \bar{q}(s), \Delta],$$

$$R^h = \hbar^2 v_3(\Delta, \hbar) = R^{\hbar}[\bar{p}(s), \bar{q}(s), \bar{f}(s), \Delta]$$

are functionals of $\bar{p}(s)$, $\bar{q}(s)$, $\bar{f}(s)$ on the segment $-1 \leqslant s \leqslant 1$, depending on the two parameters h and Δ or on h_l and h_p $(h = 0.5\ (h_l + h_p),\ \ \Delta = -0.5\ (h_r - h_l)/h)$.

To obtain the difference scheme, let us return to the initial variable x, writing $s = \Delta_i + (x - x_i)/h_i$ and using (6). The values of the pattern functions we then obtain we will denote by $\overset{(i)}{v_k}(s, h)$, $k = 1, 2, 3$. The relation (10) takes the form (4), where

$$P_i^h = P^{\hbar}[p(\bar{x}_i + s\hbar_i), q(\bar{x}_i + s\hbar_i), \Delta_i], \tag{11}$$

$$Q_i^h = Q^{\hbar}[p(\bar{x}_i + s\hbar_i), q(\bar{x}_i + s\hbar_i), \Delta_i], \tag{12}$$

$$R_i^h = R^{\hbar}[p(\bar{x}_i + s\hbar_i), q(\bar{x}_i + s\hbar_i), f(\bar{x}_i + s\hbar_i), \Delta_i]. \tag{13}$$

It is clear from the construction of the scheme (4) that the solution of the difference boundary problem

$$y_i = P_i^h y_{i-1} + Q_i^h y_{i+1} + R_i^h,\ \ 0 < i < N,\ \ y_0 = \mu_1,\ \ y_N = \mu_2, \tag{14}$$

is the same as the solution of the initial problem:

$$y_i = u\,(x_i),\ \ i = 0, 1, \ldots, N,$$

at nodal points of the arbitrary net S_N.

2. *Some properties of pattern functions*

We shall need a number of properties of the pattern functions $v_1(s,\ h)$ and $v_2(s,\ h)$ in what follows.

Lemma 1. If $\bar{q}(s) \geqslant 0$, then the pattern function $v_1(s,\ h)$ is positive in the integral $-1 < s \leqslant 1$ and $v_2(s,\ h) > 0$ in the interval $-1 \leqslant s < 1$.

This obvious statement follows from the fact that for $\bar{q}(s) \geqslant 0$ all the eigenvalues of the first boundary problem for the operator $\mathscr{L}^*$ are positive. For the same reason v_1 and v_2 are linearly independent.

Lemma 2. The pattern functions $v_1(s,\ h)$ and $v_2(s,\ h)$ satisfy the relations

1) $v_1\,(1, \hbar) = v_2\,(-1, \hbar)$, (15)

2) $v_1\,(1, \hbar) - v_1\,(\Delta, \hbar) - v_2\,(\Delta, \hbar) =$

$$= \hbar^2 \left\{ v_2\,(\Delta, \hbar) \int\limits_{-1}^{\Delta} \bar{q}\,(s)\; v_1\,(s, \hbar)\, ds + v_1\,(\Delta, \hbar) \int\limits_{\Delta}^{1} \bar{q}\,(s)\; v_2\,(s, \hbar)\, ds \right\}, \tag{16}$$

if $\bar{q}(s) \geqslant 0$.

We note that condition (15) expresses the reciprocity principle for an ordinary differential equation. It follows from Green's second formula:

$$\int\limits_{-1}^{1} (v_1 \mathscr{L}^* v_2 - v_2 \mathscr{L}^* v_1)\, ds = \frac{1}{\bar{p}\,(s)} \left(v_1 \frac{dv_2}{ds} - v_2 \frac{dv_1}{ds} \right) \Bigg|_{-1}^{1} = 0.$$

To obtain formula (16) we integrate the equation $\mathscr{L}^* v_1 = 0$ with respect to s from -1 to Δ, and the equation $\mathscr{L}^* v_2 = 0$ with respect to s from Δ to 1, and then use the conditions (7):

$$\left(\frac{1}{\bar{p}} \frac{dv_1}{ds} \right)_{s=\Delta} = 1 + \hbar^2 \int\limits_{-1}^{\Delta} \bar{q} v_1\, ds,$$

$$\left(\frac{1}{\bar{p}}\frac{dv_2}{ds}\right)_{s=\Delta} = -1 - \hbar^2 \int_{\Delta}^{1} \bar{q} v_2 \, ds. \tag{17}$$

On the segments $-1 \leqslant s \leqslant \Delta$ and $\Delta \leqslant s \leqslant 1$ Green's formula for v_1 and v_2 gives

$$\frac{1}{\bar{p}}(v_2 v_1' - v_1 v_2')\big|_{s=\Delta} = \frac{1}{\bar{p}}(v_2 v_1' - v_1 v_2')\big|_{s=-1} = v_2(-1, \hbar),$$
$$\frac{1}{\bar{p}}(v_2 v_1' - v_1 v_2')\big|_{s=\Delta} = \frac{1}{\bar{p}}(v_2 v_1' - v_1 v_2')\big| = v_1(1, \hbar).$$

Comparing these expressions, we obtain

$$v_2(\Delta, \hbar) \cdot \left(\frac{1}{\bar{p}}\frac{dv_1}{ds}\right)_{s=\Delta} - v_1(\Delta, \hbar)\left(\frac{1}{\bar{p}}\frac{dv_2}{ds}\right)_{s=\Delta} = v_1(1, \hbar).$$

By substituting the expressions (17) in this equation, we arrive at formula (16).

Our arguments still apply when the piece-wise continuous function $p(s)$ is being considered, since at its points of discontinuity the junction conditions

$$\left(\frac{1}{\bar{p}}\frac{dv_k}{ds}\right)_l = \left(\frac{1}{\bar{p}}\frac{dv_k}{ds}\right)_r, \quad (v_k)_l = (v_k)_r, \quad k = 1, 2,$$

are satisfied, where the suffixes l and r correspond to the left and right limiting values.

It is not difficult to show that Green's function $G^h(s, t)$ for the operator $\mathscr{L}^*$ for the first boundary problem can be expressed quite simply in terms of the pattern functions v_1 and v_2:

$$G^{\hbar}(s, t) = \begin{cases} \dfrac{v_1(s, \hbar)\, v_2(t, \hbar)}{v_1(1, \hbar)}, & s < t, \\[2ex] \dfrac{v_1(t, \hbar)\, v_2(s, \hbar)}{v_1(1, \hbar)}, & s > t, \end{cases}$$

since

$$\frac{1}{\bar{p}}(v_2 v_1 - v_1 v_2') = v_1(1, \hbar) = v_2(-1, \hbar).$$

3. The conservativeness of the exact scheme

Using the properties of the tabular functions which were established in Section 2 we can transform the exact difference scheme to the form (2).

It is more convenient to transform first equation (11), which from (15) we can write in the form

$$\frac{\bar{u}(1)-\bar{u}(\Delta)}{v_2(\Delta,\hbar)}-\frac{\bar{u}(\Delta)-\bar{u}(-1)}{v_1(\Delta,\hbar)}-\frac{v_1(1,\hbar)-v_1(\Delta,\hbar)-v_2(\Delta,\hbar)}{v_1(\Delta,\hbar)\,v_2(\Delta,\hbar)}\,\bar{u}(\Delta)=$$

$$=-\hbar^2\frac{v_1(1,\hbar)\,v_3(\Delta,\hbar)}{v_1(\Delta,\hbar)\,v_3(\Delta,\hbar)}.$$

Using (16) we can eliminate $v_1(1, h)$. To write down the corresponding difference equation at the point $x = x_i$ of the net, we must put $\bar{u}(s) = u(\bar{x}_i + sh_i)$, $\bar{p}(s) = p(\bar{x}_i + sh_i)$ and so on, where $\bar{x}_i = x_i - h_i \Delta_i$. Replacing u_i by y_i we can write the exact scheme in the form

$$L_h^{(p,q,f)}y_i=\frac{1}{\hbar_i}\left(\frac{\Delta y_i}{h_{i+1}B_i^h}-\frac{\nabla y_i}{h_iA_i^h}\right)-D_i^h\cdot y_i+\Phi_i^h=0, \tag{18}$$

where

$$A_i^h=\frac{\hbar_i}{h_i}\overset{i}{v}_1(\Delta_i,\hbar_i),\quad B_i^h=\frac{\hbar_i}{h_{i+1}}\overset{i}{v}_2(\Delta_i,\hbar_i), \tag{19}$$

$$D_i^h=\frac{\hbar_i}{\hbar_iA_i^h}\int_{-1}^{\Delta_i}q(\bar{x}_i+s\hbar_i)\overset{i}{v}_1(s,\hbar_i)\,ds+\frac{\hbar_i}{h_{i+1}B_i^h}\int_{\Delta_i}^{1}q(\bar{x}_i+s\hbar_i)\overset{i}{v}_2(s,\hbar_i)\,ds, \tag{20}$$

$$\Phi_i^h=\left(\hbar_iD_i^h+\frac{1}{h_iA_i^h}+\frac{1}{h_{i+1}B_i^h}\right)\cdot\hbar_i\overset{i}{v}_3(\Delta_i,\hbar_i). \tag{21}$$

Lemma 3. The exact difference scheme $L_h^{(p,q,f)}$ defined by the formulae (18) to (21) is conservative, i.e.

$$B_i^h=A_{i+1}^h\quad\text{or}\quad\hbar_i\overset{i}{v}_2(\Delta,\hbar)=\overset{i+1}{v}_1(\Delta,\hbar)\cdot\hbar_{i+1} \tag{22}$$

Thus, write

$$\overset{i}{V}_k(x,\hbar)=\overset{i}{v}_k(s,\hbar),\ x=\bar{x}_i+s\hbar_i,\ \bar{x}_i=x_i-\hbar_i\Delta_i.$$

By definition, the functions $\overset{i+1}{V}_1(x, h)$ and $\overset{i}{V}_2(x, h)$ satisfy the conditions

$$\overset{i+1}{V}_1(x_i,\hbar_{i+1})=0,\qquad \overset{i}{V}_2(x_{i+1},\hbar)=0,$$

$$\left(\frac{1}{p}\frac{d\overset{i+1}{V}_1}{dx}\right)_{x=x_i}=\frac{1}{\hbar_{i+1}},\qquad \left(\frac{1}{p}\frac{d\overset{i}{V}_2}{dx}\right)_{x=x_{i+1}}=-\frac{1}{\hbar_i}.$$

Using the second Green formula for the functions $\overset{i+1}{V}_1(x, h)$ and $\overset{i}{V}_2(x, h)$ on the segment $x_i \leqslant x \leqslant x_{i+1}$ we obtain

$$0 = \int_{x_i}^{x_{i+1}} \{\overset{i+1}{V}_1 L^{(p,q)} \overset{i}{V}_2 - \overset{i}{V}_2 L^{(p,q)} \overset{i+1}{V}\}\, dx = \overset{i+1}{V}_1 \frac{1}{p}\frac{d\overset{i}{V}_2}{dx} - \overset{i}{V}_2 \frac{1}{p}\frac{d\overset{i+1}{V}_1}{dx}\Bigg|_{x_i}^{x_{i+1}} =$$

$$= \frac{1}{\hbar_{i+1}} \overset{i}{V}_2(x_i, \hbar) - \frac{1}{\hbar_i} \overset{i+1}{V}_1 (x_{i+1}, \hbar_{i+1}),$$

i.e.

$$\hbar_i \overset{i}{V}_2 (x_i, \hbar_i) = \hbar_{i+1} \overset{i+1}{V}_1 (x_{i+1}, h_{i+1}) \quad \text{or} \quad h_{i+1} B_i^h = h_{i+1} A_{i+1}^h,$$

from which it follows that

$$B_i^h = A_{i+1}^h.$$

The fact that the scheme $L_h^{(p,q,f)}$ is conservative means that it can be written in the form

$$L_h^{(p,q,f)} y_i = \frac{1}{\hbar_i} \Delta \left(\frac{\nabla y_i}{h_i A_i^h}\right) - D_i^h \cdot y_i + \Phi_i^h.$$

It is not difficult to see too that the coefficient A_i^h depends only on the step $h_i = x_i - x_{i-1}$ and does not depend on $\hbar_i$, i.e. on h_{i+1}.

Thus

$$A_i^h = \frac{\hbar_i}{h_i} \overset{i}{V}_1 (x_i, \hbar_i) = \frac{1}{h_i} \overset{i}{\overline{V}}_1 (x_i),$$

where $\overset{i}{\overline{V}}(x)$ is the solution of the equation

$$L^{(p,q)} \overline{V}_1 = 0$$

with the initial data

$$\overset{i}{\overline{V}}_1 (x_{i-1}) = 0, \frac{1}{p} \frac{d\overset{i}{\overline{V}}_1}{dx}\Bigg|_{x=x_{i-1}} = 1.$$

4. *Exact boundary conditions of the third kind*

We now consider the boundary problem of the third kind. For simplicity we shall assume that the condition of the third kind is given only for $x = 0$, and that at the right-hand end $x = 1$ the previous condition of the first kind applies:

$$L^{(p,q,f)} u = 0, \ 0 < x < 1, \ (1/p(0))\, u'(0) - \sigma u(0) = \mu_1, \ u(1) = \mu_2. \quad (23)$$

We find the exact difference boundary condition for $x = 0$ which any solution of the problem (23) will satisfy.

We introduce the local system of co-ordinates $s = x/h_1$ at the point $x = 0$. Just as in Section 1 we introduce the pattern functions $v_1^*(s, h_1)$, $v_2(s, h_1)$, $v_3^*(s, h_1)$ in the interval $0 < s < 1$ using the conditions

$$\mathscr{L}^* v_1^* = \mathscr{L}^* v_2 = 0, \; \mathscr{L}^* v_3^* = -\bar{f}(s). \tag{24}$$

$$\left.\begin{aligned} &v_1^*(0, h_1) = 0, && \frac{1}{\bar{p}(0)} \frac{dv_1^*}{ds}(0, h_1) = 1, \\ &v_2(1, h_1) = 0, && \frac{1}{\bar{p}(1)} \frac{dv_2}{ds}(1, h_1) = -1, \\ &v_3^*(0, h_1) = 0, && v_3^*(1, h_1) = 0. \end{aligned}\right\} \tag{25}$$

We put the general solution of the equation $\mathscr{L}^* u(s) = -h_1^2 f(s)$ in the form

$$\bar{u}(s) = \frac{v_2(s, h_1)}{v_2(0, h_1)} \bar{u}(0) + \frac{v_1^*(s, h_1)}{v_1(1, h_1)} \bar{u}(1) + h_1^2 v_3^*(s, h_1). \tag{26}$$

It will be necessary for this function to satisfy the condition

$$\frac{1}{h_1 \bar{p}(0)} \frac{d\bar{u}}{ds}(0) - \sigma \bar{u}(0) = \mu_1. \tag{27}$$

Inserting (26) in (27) and using conditions (25) we see that

$$\bar{u}(0) = a_1 \bar{u}(1) + b_1,$$

where

$$\left.\begin{aligned} a_1 &= \left\{1 + h_1 \left[\sigma v_2(0, h_1) + h_1 \int_0^1 \bar{q}(s)\, v_2(s, h_1)\, ds\right]\right\}^{-1}, \\ b_1 &= h_1 \left[\mu_1 - \frac{h_1}{\bar{p}(0)} \frac{dv_3^*}{ds}(0, h_1)\right] a_1 v_2(0, h_1). \end{aligned}\right\} \tag{28}$$

Returning to the initial variable $x = sh$, we obtain the exact boundary condition in the form

$$u_0 = a_1 u_1 + b_1, \tag{29}$$

where $u_0 = u(0)$, $u_1 = u(x_1)$, $x_1 = h_1$, a_1 and b_1 are given by formulae (28), with

$$\bar{q}(s) = q(sh_1), \quad \bar{p}(s) = p(sh_1), \quad \bar{f}(s) = f(sh_1).$$

Thus the solution of the boundary problem

$$L_h^{(p,q,f)} y_i = 0, \quad 0 < i < N, \quad y_0 = a_1 y_1 + b_1, \quad y_N = \mu_2, \tag{30}$$

where $L_h^{(p,q,f)}$ is the exact scheme of Section 3 coincides at the nodal points of the difference net with the solution of the problem (23):

$$y_i = u(x_i).$$

The boundary condition

$$\frac{u'(1)}{p(1)} + \sigma u(1) = \mu_2 \tag{31}$$

corresponds to the exact boundary condition

$$u_N = a_2 u_{N-1} + b_2, \tag{32}$$

where

$$a_2 = \left\{1 + h_N \left[\sigma \overset{N}{v}_1(s, h_N) + h_N \int_{-1}^{0} q(1 + sh_N) \overset{N}{v}_1(s, h_N)\, ds\right]\right\}^{-1},$$

$$b_2 = h_N \left[\mu_2 - \frac{h_N}{p(1)} \frac{d}{ds} \overset{(N)}{v}_3(0, h_N)\right] a_2 \overset{(N)}{v}_1(0, h_N), \tag{33}$$

$v_1(s, h_N)$ is the pattern function defined on the interval $-1 < s < 0$ by the conditions (7) for $s = -1$ and $v_3(s, h_N)$ is defined on the same interval by the conditions $v_3 = 0$ for $s = -1$ and $s = 0$. The index N shows that the origin of the local co-ordinate system is taken at the point $x_N = 1$.

5. *The exact scheme* $L_h^{(p,f)}$

If $q(x) \equiv 0$, then all the pattern functions, and therefore the functionals A^h and Φ^h, are calculated with the help of a quadrature.

The exact scheme $L_h^{(p,f)}$ is of the form

$$L_h^{(p,f)} y_i = \frac{1}{\hbar_i} \Delta\left(\frac{\nabla y_i}{h_i A_i^h}\right) + \Phi_i^h, \tag{34}$$

where

$$A_i^h = \frac{\hbar_i}{h_i} \overset{i}{v}_1(0, h) = \int_{-1}^{0} p(x_i + sh_i)\, ds = \frac{1}{h_i} \int_{x_{i-1}}^{x_i} p(x)\, dx, \tag{35}$$

$$\Phi_i^h = F_i^h + \frac{1}{\hbar_i} \Delta\left(\frac{h_i}{A_i^h} \int_{-1}^{0} p(x_i + th_i)\left[\int_{-0,5}^{t} f(x_i + sh_i)\, ds\right] dt\right), \tag{36}$$

$$F_i^h = \frac{1}{h_i} \int_{x_{i-0,5}}^{x_{i+0,5}} f(x)\, dx \qquad (x_{i-0,5} = x_i - 0{,}5h_i). \tag{37}$$

Thus we have

$$v_1(s, h) = v_1(s) = \int_{-1}^{s} \bar{p}(s)\, ds, \qquad v_2(s, h) = v_2(s) = \int_{s}^{1} \bar{p}(s)\, ds,$$

$$v_3(s, h) = v_3(s) = \left\{ \int_{-1}^{1} \bar{p}(t) \left[\int_{-1}^{t} \bar{f}(\lambda)\, d\lambda \right] dt\, v_1(s) - \right.$$

$$\left. - v_1(1) \int_{-1}^{s} \bar{p}(t) \left[\int_{-1}^{t} \bar{f}(\lambda)\, d\lambda \right] dt \right\} \frac{1}{v_1(1)}.$$

It follows that

$$\Phi_i^h = \frac{1}{h_i} \left\{ \frac{1}{h_{i+1} A_{i+1}^h} \int_{x_i}^{x_{i+1}} p(x) \left[\int_{x_{i-1}}^{x} f(\xi)\, d\xi \right] dx - \right.$$

$$\left. - \frac{1}{h_i A_i^h} \int_{x_{i-1}}^{x_i} p(x) \left[\int_{x_{i-1}}^{x} f(\xi)\, d\xi \right] dx \right\}. \tag{38}$$

Let us put Φ_i^h in the form of the sum of the principal part F_i^h and a conservative ("divergent") addition:

$$\Phi_i^h = F_i^h + \frac{1}{h_i} \Delta \widetilde{F}_i^h.$$

To do this we write the terms in the curly brackets in formula (38) as follows:

$$\frac{1}{A_{i+1}^h} \int_{x_i}^{x_{i+1}} p(x) \left[\int_{x_{i-1}}^{x} f(\xi)\, d\xi \right] dx = \frac{1}{A_{i+1}^h} \int_{x_i}^{x_{i+1}} p(x) \left[\int_{x_{i-1}}^{x_{i+0,5}} f(\xi)\, d\xi + \right.$$

$$\left. + \int_{x_{i+0,5}}^{x_i} f d\xi \right] dx = h_{i+1} \int_{x_{i-1}}^{x_{i+0,5}} f dx + \frac{1}{A_{i+1}^h} \int_{x_i}^{x_{i+1}} p(x) \left[\int_{x_{i+0,5}}^{x} f d\xi \right] dx;$$

$$\frac{1}{A_i^h} \int_{x_{i-1}}^{x_i} p(x) \left[\int_{x_{i-1}}^{x} f d\xi \right] dx = h_i \int_{x_{i-1}}^{x_{i-0,5}} f dx + \frac{1}{A_i^h} \int_{x_{i-1}}^{x_i} p(x) \left[\int_{x_{i-0,5}}^{x} f d\xi \right] dx.$$

As a result, we obtain expression (36) for Φ^h. The second term in (36) is of the second order of smallness as $h^* \to 0$ in the class of sufficiently smooth functions $p(x)$ and $f(x)$:

$$\Phi_i^h - F_i^h = O(h^{*2}) \qquad (h^* = \| h_i \|_0 = \max_{1<i\leqslant N} h_i).$$

If we set $f = 0$, then we obtain the best canonical scheme

$$L_h^{(p)} y_i = L_h^{(p,0)} y_i = \frac{1}{\hbar_i} \Delta \left(\frac{\nabla y_i}{h_i A_i^h} \right), \quad A_i^h = \int_{-1}^{0} p(x_i + s h_i)\, ds, \tag{39}$$

which is the exact scheme for the equation $L^{(p)} u = 0$.

6. *Determination of the pattern functions*

In the general case, when $q(x) \not\equiv 0$, it is not possible to express the pattern functions directly using a quadrature. However, since $v_1(s, \hbar)$, $v_2(s, \hbar)$ and $v_3(s, \hbar)$ are analytic in $\hbar^2$, it is natural to look for these functions in the form of a power series in $\hbar^2$:

$$v_j(s, \hbar) = \sum_{k=0}^{\infty} v_j^{(k)}(s)\, \hbar^{2k}, \quad j = 1, 2, 3. \tag{40}$$

The coefficients of the series $v_j^{(k)}(s)$ satisfy the equations

$$\begin{gathered} \frac{d}{ds}\left[\frac{1}{\bar{p}(s)} \frac{dv_j^{(0)}}{ds} \right] = 0, \quad j = 1, 2; \quad \frac{d}{ds}\left[\frac{1}{\bar{p}(s)} \frac{dv_3^{(0)}}{ds} \right] = -\bar{f}(s), \\ \frac{d}{ds}\left[\frac{1}{\bar{p}(s)} \frac{dv_j^{(k)}(s)}{ds} \right] = \bar{q}(s)\, v_j^{(k-1)}(s), \quad j = 1, 2, 3, \quad k = 1, 2, \ldots, \end{gathered} \tag{41}$$

with the following additional conditions

$$\begin{gathered} v_1^{(k)}(-1) = v_2^{(k)}(1) = v_3^{(k)}(\pm 1) = 0, \quad k = 0, 1, 2, \ldots, \\ \frac{1}{\bar{p}(-1)} \frac{dv_1^{(0)}}{ds}(-1) = 1, \quad \frac{1}{\bar{p}(1)} \frac{dv_2^{(0)}}{ds}(1) = -1, \\ \frac{dv_1^{(k)}}{ds}(-1) = \frac{dv_2^{(k)}}{ds}(1) = 0, \quad k = 1, 2, \ldots \end{gathered} \tag{42}$$

This gives

$$v_1^{(0)}(s) = \int_{-1}^{s} \bar{p}(t)\, dt, \quad v_2^{(0)}(s) = \int_{s}^{1} \bar{p}(t)\, dt,$$

$$v_1^{(k)}(s) = \int_{-1}^{s} \bar{p}(t) \left[\int_{-1}^{t} \bar{q}(\lambda)\, v_1^{(k-1)}(\lambda)\, d\lambda \right] dt, \qquad k = 1, 2, \ldots,$$

$$v_2^{(k)}(s) = \int_{s}^{1} \bar{p}(t) \left[\int_{t}^{1} \bar{q}(\lambda)\, v_2^{(k-1)}(\lambda)\, d\lambda \right] dt, \qquad k = 1, 2, \ldots,$$

$$v_3^{(k)}(s) = \frac{1}{v_1^{(0)}(1)}\left\{v_1^{(0)}(s)\int_{-1}^{1}\bar{p}(t)\left[\int_{-1}^{t}\omega^{(k)}(\zeta)\,d\zeta\right]dt - \right.$$
$$\left. - v_1^{(0)}(1)\int_{-1}^{s}\bar{p}(t)\left[\int_{-1}^{t}\omega^{(k)}(\zeta)\,d\zeta\right]dt\right\},$$

where

$$\omega^{(0)}(s) = -\bar{f}(s),\ \omega^{(k)}(s) = \bar{q}(s)\,v_3^{(k-1)}(s) \quad \text{for } k = 1, 2, \ldots .$$

2. Difference schemes of a high order of accuracy

1. Truncated difference schemes

In Section 1, Paragraph 6, we obtained the power series expansion in h^2 for the pattern functions $v_j(s, h)$, $j = 1, 2, 3$ with the help of which the coefficients of the exact scheme are calculated.

If, in the expansion

$$v_j(s, \hbar) = \sum_{k=0}^{\infty} v_j^{(k)}(s)\,\hbar^{2k}, \qquad j = 1, 2, 3, \tag{1}$$

we restrict ourselves to a finite number of terms and take the polynomials

$$\Pi_j^{(m)}(s, \hbar) = \sum_{k=0}^{m} v_j^{(k)}(s)\cdot h^{2k}, \tag{2}$$

for the pattern functions, then we obtain the difference scheme

$$L_h^{(p,q,f)}y_i = \frac{1}{\hbar_i}\Delta\left(\frac{\nabla y_i}{h_i A_i^h}\right) - D_i^h\cdot y_i + \Phi_i^h, \tag{3}$$

whose coefficients A_i^h, D_i^h, Φ_i^h are calculated from the same formulae as the coefficients A_i^h, D_i^h, Φ_i^h of the exact scheme, putting the polynomials $\Pi_j^{(m)}(s, \hbar)$ in place of the functions $v_i(s, \hbar)$.

A difference scheme constructed in this way will be called a truncated difference scheme of the mth rank.

The following account is devoted to finding the order of accuracy of a truncated scheme.

2. A comparison of the solutions of difference problems

Let us compare the solutions of the two difference boundary problems

$$L_h y_i = \frac{1}{\hbar_i} \Delta \left(\frac{\nabla y_i}{h_i A_i^h} \right) - D_i^h y_i = -\Phi_i^h, \; 0 < i < N, \; y_0 = \mu_1, \; y_N = \mu_2, \quad (4)$$

$$\widetilde{L}_h \widetilde{y}_i = \frac{1}{\hbar_i} \Delta \left(\frac{\nabla \widetilde{y}_i}{h_i \widetilde{A}_i^h} \right) - \widetilde{D}_i^h \widetilde{y}_i = -\widetilde{\Phi}_i^h, \; 0 < i < N, \; \widetilde{y}_0 = \mu_1, \; \widetilde{y}_N = \mu_2, \quad (5)$$

whose coefficients satisfy the conditions

$$\begin{gathered} 0 < M_1 \leqslant A_i^h \leqslant M_2, \; 0 < M_1 \leqslant \widetilde{A}_i^h \leqslant M_2, \\ 0 \leqslant D_i^h \leqslant M_3, \; 0 \leqslant \widetilde{D}_i^h \leqslant M_3, \end{gathered} \quad (6)$$

where M_1, M_2, M_3 are positive constants which do not depend on the net. By analogy with [3], where a uniform net was considered, it is not difficult to show that Green's difference function of the operator L_h (or $\widetilde{L}_h$) is bounded:

$$0 \leqslant G_{ik} \leqslant M \quad (7)$$

and has bounded first difference ratios

$$\begin{gathered} \frac{1}{h_{i+1}} | G_{i+1,k} - G_{i,k} | \leqslant M, \qquad \frac{1}{h_{k+1}} | G_{i,k+1} - G_{i,k} | \leqslant M, \\ i, k = 0, 1, 2, \ldots, N-1, \end{gathered} \quad (8)$$

where M is a positive constant independent of the net.

We use the notation

$$\| \psi \|_0 = \max_{0 \leqslant i \leqslant N} | \psi_i |, \; \| \psi \|_1 = \sum_{i=1}^{N-1} | \psi_i | \hbar_i, \qquad \| \psi \|_1^* = \sum_{i=1}^{N} | \psi_i | h_i, \quad (9)$$

where ψ_i is any net function.

Lemma 4. If the conditions (6) are satisfied and, in addition, Φ_i^h, μ_1 and μ_2 are bounded, then

$$\left| \frac{\Delta y_i}{h_{i+1}} \right| \leqslant M, \quad i = 0, 1, 2, \ldots, N-1, \quad (10)$$

where M is a positive constant, independent of the net.

Thus the solution of the problem (4) can be found using Green's difference function

$$y_i = \sum_{k=1}^{N-1} G_{ik}\Phi_k^h \cdot \hbar_k - A_1 \frac{G_{i,1}}{h_1}\mu_1 - A_N \frac{G_{i,N-1}}{h_N}\mu_2, \tag{11}$$

$$\frac{\Delta y_i}{h_{i+1}} = \sum_{k=1}^{N-1} \frac{G_{i+1,k} - G_{i,k}}{h_{i+1}} \Phi_k^h \cdot \hbar -$$

$$- A_1 \frac{G_{i+1,1} - G_{i,1}}{h_1 h_{i+1}} \mu_1 - A_N \frac{G_{i+1,N-1} - G_{i,N-1}}{h_N h_{i+1}} \mu_2. \tag{11'}$$

Green's difference function is

$$G_{ik} = \begin{cases} \dfrac{\alpha_i \beta_k}{\alpha_N}, & i < k, \\ \dfrac{\alpha_k \beta_i}{\alpha_N}, & i > k, \end{cases} \tag{12}$$

where α_i and β_i are solutions of the equation $L_h v_i = 0$ with the conditions

$$\alpha_0 = 0, \frac{1}{A_1^h} \frac{\Delta \alpha_0}{h_1} = 1, \quad \beta_N = 0, \quad \frac{1}{A_N} \frac{\Delta \beta_{N-1}}{h_N} = -1. \tag{13}$$

It follows that

$$\frac{1}{h_1 h_{i+1}} (G_{i+1,1} - G_{i,1}) = \frac{1}{h_1} \frac{\Delta \alpha_0}{\alpha_N} \frac{\Delta \beta_i}{h_{i+1}} = \frac{A_1^h}{\alpha_N} \frac{\Delta \beta_i}{h_{i+1}} = O(1)$$

and similarly

$$\frac{1}{h_{i+1} h_N} (G_{i+1,N-1} - G_{i,N-1}) = O(1).$$

It follows from (11′) that $\Delta y_i / h_{i+1}$ is bounded.

Lemma 5. If the conditions of Lemma 4 are satisfied, then

$$\| y - \widetilde{y} \|_0 \leqslant M \{ \| A^h - \widetilde{A}^h \|_1 + \| D^h - D^h \|_1 + \| \Phi^h - \widetilde{\Phi}^h \|_1 \}, \tag{14}$$

where y is a solution of problem (4), $\widetilde{y}$ is a solution of problem (5) and M is a positive constant independent of the net.

To prove the lemma, we form an equation for the difference $z = y_i - \widetilde{y}_i$:

$$\widetilde{L}_h z_i = - \left\{ (\Phi_i^h - \widetilde{\Phi}_i^h) + \frac{1}{\hbar_i} \Delta \left(\left(\frac{1}{A_i^h} - \frac{1}{\widetilde{A}_i^h} \right) \frac{\nabla y_i}{h_i} \right) - \right.$$

$$\left. - (D_i^h - \widetilde{D}_i^h) y_i \right\} = - \Psi_i, \quad z_0 = z_N = 0.$$

This gives

$$z_i = \sum_{k=1}^{N-1} \hbar_k \tilde{G}_{ik} \{-(D_k^h - \tilde{D}_k^h) y_k + (\Phi_k^h - \tilde{\Phi}_k^h)\} -$$

$$- \sum_{k=0}^{N-1} \frac{\tilde{G}_{i,\,k+1} - \tilde{G}_{i,k}}{h_{k+1}} \left(\frac{1}{A_{k+1}^h} - \frac{1}{\tilde{A}_{k+1}^h}\right) \frac{\Delta y_k}{h_{k+1}} \cdot h_{k+1},$$

since

$$\sum_{i=1}^{N-1} \tilde{G}_{ik} \Delta v_k = - \sum_{k=0}^{N-1} (\tilde{G}_{i,\,k+1} - \tilde{G}_{i,k}) \cdot v_{k+1} \qquad (\tilde{G}_{ik} = 0 \ \ \textbf{for} \ k = 0, N).$$

To obtain the inequality (14), we use the fact that the functions y_i, $\tilde{G}_{ik}$ and their first difference ratios are bounded.

Returning now to exact and truncated difference schemes, from Lemma 5 we have

$$\| \overset{(m)}{y^h} - u \|_0 \leqslant M \{ \| A^h - \overset{(m)}{A^h} \|_1^* + \| D^h - \overset{(m)}{D^h} \|_1 + \| \Phi^h - \overset{(m)}{\Phi^h} \|_1 \}. \qquad (15)$$

where $u = u(x)$ is a solution of the initial problem, and $\overset{(m)}{y^h}$ is a solution of the difference boundary problem

$$\begin{gathered} \overset{m}{L}{}_h^{(p,q,f)} \overset{(m)}{y_i^h} = 0, \quad 0 < i < N, \\ \overset{(m)}{y_0^h} = \mu_1, \quad \overset{(m)}{y_N^h} = \mu_2, \end{gathered} \qquad (16)$$

where $\overset{m}{L}{}_h^{(p,q,f)}$ is the truncated scheme (3) of the mth rank.

To estimate the error in the coefficients of the scheme on the right-hand side of formula (15), we must evaluate the difference

$$\begin{aligned} v_j(s, \hbar) - \Pi_j^{(m)}(s, \hbar) &= \hbar^{2m+2} (v_j^{(m+1)}(s) + \hbar^2 v_j^{(m+2)}(s) + \ldots) = \\ &= \hbar^{2m+2} \Omega_j^{(m+1)}(s, \hbar). \end{aligned} \qquad (17)$$

3. Estimate of the remainder term in the expansion of the pattern functions

Let us turn now to the proof of the boundedness, for any $m \geqslant 0$, of the functions

$$\Omega_j^{(m+1)}(s, \hbar) = \sum_{k=0}^{\infty} v_j^{(m+1+k)}(s)\, \hbar^{2k}, \quad j = 1, 2, 3, \; m \geqslant 0. \tag{18}$$

First we shall show that the inequalities for $\Omega_1^{(m+1)}(s, h)$ and $\Omega_2^{(m+1)}(s, h)$

$$\begin{aligned} 0 &\leqslant \Omega_1^{(m+1)}(s, \hbar) \leqslant M \frac{[\varkappa(1+s)]^{2m+1}}{(2m+1)!}, \\ 0 &\leqslant \Omega_2^{(m+1)}(s, \hbar) \leqslant M \frac{[\varkappa(1-s)]^{2m-1}}{(2m+1)!}, \end{aligned} \tag{19}$$

are satisfied, where $\kappa = \sqrt{(M_2 M_3)}$, and M is some positive constant depending on M_1, M_2 and M_3 only.

Thus using conditions (41), (42) of Section 1 we have, for $v_j^{(k)}(s)$

$$\frac{d}{ds}\left[\frac{1}{\bar{p}(s)} \frac{d\Omega_j^{(m+1)}}{ds}\right] = \bar{q}(s)\, \Omega_j^{(m)}(s, \hbar),$$

$$\Omega_1^{(m+1)}(-1, \hbar) = \frac{d\Omega_1^{(m+1)}}{ds}(-1, \hbar) = 0, \quad \Omega_2^{(m+1)}(1, \hbar) = \frac{d\Omega_2^{(m+1)}}{ds}(1, \hbar) = 0.$$

This gives

$$\Omega_1^{(m+1)}(s, \hbar) = \int_{-1}^{s} \bar{p}(t) \left[\int_{-1}^{t} \bar{q}(\lambda)\, \Omega_1^{(m)}(\lambda, \hbar)\, d\lambda\right] dt,$$

$$\Omega_2^{(m+1)}(s, \hbar) = \int_{s}^{1} \bar{p}(t) \left[\int_{t}^{1} \bar{q}(\lambda)\, \Omega_2^{(m)}(\lambda, \hbar)\, d\lambda\right] dt.$$

Then, using the conditions $M_1 \leqslant \bar{p} \leqslant M_2$, $\bar{q} \leqslant M_3$ we have

$$\begin{aligned} \Omega_1^{(m+1)}(s, \hbar) &\leqslant \varkappa^2 \int_{-1}^{s}\int_{-1}^{t} \Omega_1^{(m)}(\lambda, \hbar)\, d\lambda\, dt, \\ \Omega_2^{(m+1)}(s, \hbar) &\leqslant \varkappa^2 \int_{s}^{1}\int_{t}^{1} \Omega_2^{(m)}(\lambda, \hbar)\, d\lambda\, dt. \end{aligned} \tag{20}$$

It is not difficult to show that for $\Omega_j^{(0)}(s, \hbar) = v_j(s, h)$

$$\Omega_1^{(0)}(s, \hbar) \leqslant M\varkappa(1+s), \qquad \Omega_2^{(0)}(s, \hbar) \leqslant M\varkappa(1-s). \tag{21}$$

(19) follows from (20) and (21).

Let us turn now to the function

$$v_3(s, \hbar) = \Pi_3^{(m)}(s, h) + \hbar^{2m+2}\Omega_3^{(m+1)}(s, h), \tag{22}$$

where $\Pi_3^{(m)}(s, h) = \sum_{k=0}^{m} v_3^{(k)}(s)\hbar^{2k}$ is an mth degree polynomial in $\zeta = \hbar^2$.

Introducing Green's function

$$G^{\hbar}(s, t) = \begin{cases} \dfrac{v_1(s, \hbar)\, v_2(t, \hbar)}{v_1(1, \hbar)}, & s < t, \\ \dfrac{v_1(t, \hbar)\, v_2(s, \hbar)}{v_1(1, \hbar)}, & s > t, \end{cases} \tag{23}$$

we write the solution of the boundary problem

$$\mathscr{L}^* v_3 = -\bar{f}(s), \quad v_3(\pm 1, \hbar) = 0$$

in the form

$$v_3(s, \hbar) = \int_{-1}^{1} G^{\hbar}(s, t)\, \bar{f}(t)\, dt. \tag{24}$$

Then, since the function $1/v_1(1, h)$ is analytic in some region $|\zeta| \leqslant \zeta_0$ of the complex variable $\zeta = \hbar^2$, it is easy to see that the function $\Omega_3^{(m+1)}(s, h)$ is analytic for sufficiently small $\hbar \leqslant \bar{h}_0$. It is therefore bounded:

$$|\Omega_3^{(m+1)}(s, \hbar)| \leqslant M, \quad \hbar \leqslant \bar{h}_0, \tag{25}$$

where M is a positive constant independent of $\hbar$ and m.

Thus the estimate of the form (25) applies to all the pattern functions.

4. The accuracy of a truncated scheme

To determine the accuracy of a truncated scheme, i.e. to evaluate the norm $\|\overset{(m)}{y}{}^h - u\|_1$, we first find the magnitude of the error caused by substituting the pattern functionals

$$A^h[\bar{p}(s), \bar{q}(s), \Delta], \ \Phi^h[\bar{p}(s), \bar{q}(s), \bar{f}(s), \Delta], \ D^h[\bar{q}(s), \bar{p}(s), \Delta]$$

of the exact scheme by the functionals $\overset{(m)}{A^h}$, $\overset{(m)}{D^h}$, $\overset{(m)}{\Phi^h}$ of the truncated scheme.

Let us first consider the difference

$$A^h - \overset{(m)}{A^h} = \frac{\hbar}{h}\,[v_1(\Delta, \hbar) - \Pi_1^{(m)}(\Delta, \hbar)] = \hbar^{2m+2}\Omega_1^{(m+1)}(\Delta, h). \tag{26}$$

It follows from the estimate in the previous section and from condition (H) (Section 1, Paragraph 1) that

$$|A^h - \overset{(m)}{A^h}| \leqslant M\,(h^*)^{2m+2}, \tag{27}$$

where M is a positive constant independent of the net and of m.

The difference $D^h - \overset{(m)}{D}{}^h$ can be put in the form

$$\begin{aligned} D^h - \overset{(m)}{D^h} = \\ = \frac{\hbar^{2m+2}}{v_1(\Delta, \hbar)}\left[\int_{-1}^{\Delta} \bar{q}(s)\,\Omega_1^{(m+1)}(s, \hbar)\,ds - \frac{\Omega_1^{(m+1)}(\Delta, \hbar)}{\Pi_1^{(m)}(\Delta, \hbar)}\int_{-1}^{\Delta} \bar{q}(s)\,\Pi_1^{(m)}(s, h)\,ds\right] + \\ + \frac{\hbar^{2m+2}}{v_2(\Delta, \hbar)}\left[\int_{\Delta}^{1} \bar{q}(s)\,\Omega_2^{(m+1)}(s, \hbar)\,ds - \frac{\Omega_2^{(m+1)}(\Delta, h)}{\Pi_2^{(m)}(\Delta, h)}\int_{\Delta}^{1} \bar{q}(s)\Pi_2^{(m)}(s, h)\,ds\right]. \end{aligned} \tag{28}$$

Because of the lower limit $p(x) \geqslant M_1 > 0$ we have $v_j(\Delta, h) \geqslant M_1 \geqslant 0$, $\Pi_j^{(m)}(\Delta, h) \geqslant M_1 > 0$, $j = 1, 2$.

Then, since $\Omega_j^{(m+1)}(s, \hbar) \leqslant M_0 (j = 1, 2)$, $\bar{q} \leqslant M_3$ we have

$$|D^h - \overset{(m)}{D^h}| \leqslant \frac{4M_0M_3}{M_1}(h^*)^{2m+2} = M\cdot(h^*)^{2m+2} \tag{29}$$

Similarly, we obtain an estimate of the error $\Phi^h - \overset{(m)}{\Phi}{}^h$ where

$$\Phi^h = \left(\hbar^2 D^h + \frac{1}{v_1(\Delta, \hbar)} + \frac{1}{v_2(\Delta, \hbar)}\right) v_3(\Delta, h),$$

$$\overset{(m)}{\Phi^h} = \left(\hbar^2 \overset{(m)}{D^h} + \frac{1}{\Pi_1^{(m)}(\Delta, \hbar)} + \frac{1}{\Pi_2^{(m)}(\Delta, \hbar)}\right)\Pi_3^{(m)}(\Delta, h).$$

As a result, we arrive at an estimate of type (27) or (29):

$$|\Phi^h - \overset{(m)}{\Phi^h}| \leqslant M\cdot(h^*)^{2m+2}, \quad h^* \leqslant \bar{h}_0. \tag{30}$$

where the coefficients $\bar{p}(s)$, $\bar{q}(s)$ and $\bar{f}(s)$ are arbitrary piece-wise continuous functions.

The following theorem follows from (27), (22), (30) and Lemma 4:

Theorem. A truncated difference scheme of the mth rank ($m \geqslant 0$) is of the $(2m+2)$th order of accuracy as $\|h\|_1 = h^* \to 0$ in the class $Q^{(0)}$ of piece-wise continuous functions $p(x)$, $q(x)$, $f(x)$ on any sequence of difference nets satisfying the condition

$$0 < C_1 \leqslant \frac{h_{i+1}}{h_i} \leqslant C_2, \tag{H}$$

or, in other words,

$$\| \overset{(m)}{y^h} - u \|_0 \leqslant M \| h \|_0^{2m+2}, \quad \| h \|_0 \leqslant \bar{h}_0, \tag{31}$$

where C_1, C_2, M and $\bar{h}_0$ are positive constants independent of m and the choice of net.

Notes 1. The theorem is stated for the first boundary problem, although it can also be extended to the case of the third boundary problem (Section 5). 2. The simplest truncated scheme is that of zero rank giving second order accuracy in the class $Q^{(0)}$ for any nets satisfying (H).

Its pattern functionals are given by the formulae

$$\overset{(0)}{A}[\bar{p}(s)] = \frac{\hbar}{h} \int_{-1}^{\Delta} \bar{p}(s)\, ds = v_1^{(0)}(\Delta) \frac{\hbar}{h}, \quad v_1^{(0)}(s) = \int_{-1}^{s} \bar{p}(s)\, ds, \tag{32}$$

$$\overset{(0)}{D}[\bar{p}(s), \bar{q}(s), \Delta] = \frac{1}{v_1^{(0)}(\Delta)} \int_{-1}^{\Delta} \bar{q}(s)\, v_1^{(0)}(s)\, ds + \frac{1}{v_2^{(0)}(\Delta)} \int_{\Delta}^{1} \bar{q}(s)\, v_2^{(0)}(s)\, ds, \tag{33}$$

$$\overset{(0)}{\Phi}{}^h [\bar{p}(s), \bar{q}(s), \bar{f}(s), \Delta] = \left(\hbar^2 \overset{0}{D} + \frac{1}{v_1^{(0)}(\Delta)} + \frac{1}{v_2^{(0)}(\Delta)} \right) v_3^{(0)}(\Delta), \tag{34}$$

where

$$v_2^{(0)}(\Delta) = \int_{\Delta}^{1} \bar{p}(s)\, ds,$$

$$v_3^{(0)}(\Delta) = \frac{1}{v_1^{(0)}(1)} \left\{ v_1^{(0)}(\Delta) \int_{-1}^{1} \bar{p}(t) \left[\int_{-1}^{t} \bar{f}(\lambda)\, d\lambda \right] dt - v_1^{(0)}(1) \int_{-1}^{\Delta} \bar{p}(t) \left[\int_{-1}^{t} \bar{f}(\lambda)\, d\lambda \right] dt \right\}.$$

The functionals $\overset{(0)}{A}$ and $\overset{(0)}{D}$ do not depend on h, i.e. are canonical (see [1]; [2]; [3]).

5. *Truncated difference boundary conditions*

Let us now consider the boundary condition of the 3rd kind

$$lu = \frac{u'(0)}{p(0)} - \sigma u(0) = \mu_1 \tag{35}$$

and the corresponding boundary problem

$$L^{(p,q,f)} u = 0, \; lu = \mu_1, \; u(1) = \mu_2. \tag{36}$$

In Section 4, Paragraph 1 it was shown that the exact boundary condition for $x = 0$ has the form

$$l_h y = \frac{y_1 - y_0}{h_1 A_1^h} - y_0 \left[\sigma + \frac{h_1}{A_1^h} \int_0^1 q(sh_1) \overset{(0)}{v_2}(s, h_1)\, ds \right] = \bar{\mu}_1, \tag{37}$$

$$\bar{\mu}_1 = \mu_1 - \frac{h_1}{p(0)} \frac{dv_3^*}{ds}(0, h_1), \; A_1^h = \overset{(0)}{v_2}(0, h_1). \tag{38}$$

Replacing the pattern functions $v_2(s, h_1)$ and $v_3^*(s, h_1)$ in (37) by the polynomials $\Pi_2^{(m)}(s, h_1)$ and $\Pi_3^{*(m)}(s, h_1)$ of the $2m$th degree in h, we obtain the truncated boundary condition of the mth rank

$$\overset{m}{l_h} y = \overset{m}{\bar{\mu}_1}.$$

It is not difficult to see that

$$\overset{m}{\bar{\mu}_1} - \bar{\mu}_1 = O(h_1^{2m+3}), \qquad \overset{m}{l_h} u - l_h u = O(h_1^{2m+2}).$$

It follows that the solution of the difference boundary problem

$$\overset{m}{L_h}{}^{(p,q,f)} y_i = 0, \quad 0 < i < N, \quad \overset{m}{l_h} y = \overset{m}{\bar{\mu}_1}, \quad y_N = \mu_2,$$

with the truncated scheme $\overset{m}{L_h}{}^{(p,q,h)}$ and the truncated boundary condition of the mth rank is of the $(2m + 2)$th order of accuracy relative to the solution of the boundary problem (36).

The truncated condition of zero rank is of the form

$$y_0 = \nu_1 y_1 + \eta_1,$$

where

$$\nu_1 = \left\{ 1 + h_1 A_1 \left[\sigma + \frac{1}{h_1 A_1} \int_0^{h_1} q(x) \left(\int_0^x p(\xi)\, d\xi \right) dx \right] \right\}^{-1}, \quad A_1 = \frac{1}{h_1} \int_0^{h_1} p(x)\, dx,$$

$$\eta_1 = \left[\mu_1 - \frac{1}{h_1 A_1} \int_0^{h_1} p(x) \left(\int_0^x f(\xi)\, d\xi \right) dx \right] h A_1 \nu_1.$$

This condition can be replaced by one which is equivalent to it as far as accuracy is concerned, but is rather simpler:

$$\nu_1 = \{1 + h_1 p_0 (\sigma + 0{,}5\, hq_0)\}^{-1}, \quad \eta_1 = hp_0\nu_1 (\mu_1 - 0{,}5hq_0).$$

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