ON THE NUMERICAL SOLUTION OF EQUATIONS IN GAS DYNAMICS WITH VARIOUS TYPES OF VISCOSITY*

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We consider finite-difference schemes with "through computation" (i.e. which do not exclude the lines of discontinuity) for the equations of gas dynamics for one-dimensional isentropic motion of a gas with various types of viscosity. The progressive difference wave is defined. It is established that the difference equations can be solved in the form of such a wave. The requirement that the profile of the progressive wave shall be monotonic enables us to obtain a condition for the choice of the viscosity coefficient.

I.

1. The equations of the one-dimensional isentropic motion of a gas in Lagrange variables have the form

$$v_t + (p+q)_x = 0, \quad \theta_t = v_x, \quad (E + 0.5v^2)_t + [(p+q)^v]_x = 0, \quad p\theta = (\gamma - 1)E, \qquad (1)$$

where q is the viscosity [1], p is the pressure, v the velocity, θ the specific volume, E the internal energy, f_x, f_t the partial derivatives with respect to x and t, and $\gamma = c_p/c_v$.

Let us consider the problem of the motion of a stationary shock wave, which is spreading with a constant velocity D. Then $p(+\infty) = p_1 = 0$; $p(-\infty) = p_2$, $\theta(+\infty) = \theta_1$, $\theta(-\infty) = \theta_2$, $v(-\infty) = v_2$, $v(+\infty) = v_1 = 0$. We shall look for the solution as a function of s, $f = f(s)$, where $s = x - Dt$.

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The problem reduces to the solution of a system of ordinary differential equations. We can easily find the integral of this system

$$q\theta = 0.5(\gamma+1)D^2(\bar{\theta}_1-\theta)(\theta-\bar{\theta}_2), \quad q(+\infty)=0. \tag{2}$$

The requirements on $q(s)$ are: 1) the system of ordinary differential equations to which our problem has led must have a continuous solution; 2) the effect of $q(s)$ must be negligibly small outside the shock layer and in the region of the rarefied wave; 3) when the dimensions of the region of motion are large in comparison with the thickness of the shock layer, Hugoniot's conditions must be satisfied.

Generally speaking, q can be a function of v, p, θ, E and their derivatives. We consider here the following expression for q:

$$\theta q = -0.5\,\nu\,|v_x|^{\mu}\,(v_x - \varkappa\,|v_x|), \quad \nu = \text{const}. \tag{3}$$

When $\mu = 1$, $\kappa = 0$ we have Neumann's viscosity [1], which does not satisfy the second of the requirements. When $\mu = 1$, $\kappa = 1$ we have a viscosity which satisfies the second of the requirements (see [2]). When $\mu = 0$ we have a linear viscosity. We shall assume that $0 \leqslant \mu \leqslant 1$.

2. Let us find the spread of the front of the shock wave caused by the viscosity. In the zone of the shock wave $v_x < 0$, and so $\theta_q = \nu|v_x|^{1+\mu}$, $\kappa = 1$. For our problem we have $\theta q = \nu D^{1+\mu}(\theta')^{1+\mu}$. We introduce the new function $\lambda(s)$ with the formula $\theta = 0.5(\bar{\theta}_1 + \bar{\theta}_2) + 0.5 \times \Delta\theta \times \lambda(s)$ where $\Delta\theta = \bar{\theta}_1 - \bar{\theta}_2$. If $\theta = \bar{\theta}_1$ then $\lambda(s) = 1$, if $\theta = \bar{\theta}_2$ then $\lambda(s) = -1$.

Using (2) we easily find

$$\lambda'(s) = a\,[1-\lambda^2(s)]^{\sigma}, \quad \sigma = \frac{1}{1+\mu}, \quad a = \left[0.5\,\frac{\gamma+1}{\nu}\,(0.5\cdot D\cdot\Delta\theta)^{1-\mu}\right]^{\sigma}. \tag{4}$$

Let s_2 denote the smallest s for which $\lambda(s) = 1$, and let s_1 denote the largest s for which $\lambda(s) = -1$. We shall call $L = s_2 - s_1$ the width of the shock layer. Integrating (4) we find

$$I = aL, \text{ where } I = \int_{-1}^{1} \frac{d\lambda}{(1-\lambda^2)^{\sigma}} = \sqrt{\pi}\,\frac{\Gamma(1-\sigma)}{\Gamma\left(\frac{3}{2}-\sigma\right)}.$$

When $\mu = 1$ $(\sigma = 0.5)$ we have $L = 2\nu/(\gamma+)^{0.5}\pi$. If we put $\nu = \nu_0 \times h^{1+\mu}$ and $L = n \times h$ where n is an integer, we obtain

$$\nu_0 = 0.5\,(\gamma+1)\,(0.5\cdot D\cdot\Delta\theta)^{1-\mu} n^{1+\mu} I^{-1-\mu}. \tag{5}$$

From this formula we can see that ν_0 is independent of the force of the shock wave only when $\mu = 1$. In this case we have

$$\nu_0 = 0.5\,(\gamma + 1)\,\frac{n^2}{\pi^2}\,. \tag{6}$$

If $0 < \mu \leqslant 1$, then L is finite. Further, if $\mu = 0$, then $L = \infty$. In this case it is convenient to introduce the concept of the effective width of the shock layer.

3. Thus, let $\mu = 0$. By the effective width L_ϵ of the shock layer we shall mean the difference $L_\epsilon = s_2 - s_1$ where s_1 is the largest s for which $\theta(s) - \overline{\theta}_2 = 0.5 \times \Delta\theta \times [1 + \lambda(s)] = \epsilon \times \Delta\theta$; s_2 is the smallest s for which $\theta(s) - \overline{\theta}_1 = -\,0.5 \times \Delta\theta[\,1 - \lambda(s)\,] = -\,\epsilon\,\Delta\theta$ where ϵ is given, $0 < \epsilon < 1$. Then $aL_\epsilon = I_\epsilon = \int_{(1-2\epsilon)}^{(1+2\epsilon)} (1 - \lambda^2)^{-1} d\lambda$ where $a = 0.25/\nu(\gamma+1)D \times \Delta\theta$. Writing $L_\epsilon = n \times h$ and $\nu = \nu_0 h$ we find

$$\nu_0 \simeq 0.25\cdot(\gamma + 1)\cdot D\cdot\Delta\theta\cdot n\cdot \ln^{-1}\left(\frac{1}{\varepsilon}\right). \tag{7}$$

II.

1. To write down the difference equations of gas dynamics we select a rectangular space-time net with steps $\Delta x = h$ and $\Delta t = \tau$. We shall use the "cross" difference scheme, which can be written in the form

$$\begin{gathered}
v_i^{j+1/2} - v_i^{j-1/2} = \gamma_0\,[(p+q)^j_{i-1/2} - (p+q)^j_{i+1/2}], \quad \gamma_0 = \frac{\tau}{h}\,, \\
\theta^{j+1}_{i+1/2} - \theta^j_{i+1/2} = \gamma_0 (v^{j+1/2}_{i+1} - v^{j+1/2}_i) \\
(\text{or} \quad h\cdot\theta^{j+1}_{i+1/2} = r^{j+1}_{i+1} - r^{j+1}_i, \quad r^{j+1}_i = r^j_i + \tau v^{j+1/2}_i), \\
E^{j+1}_{i+1/2} - E^j_{i+1/2} = 0{,}5\,[(v^{j-1/2}_{i+1})^2 - (v^{j+1/2}_{i+1})^2] + \gamma_0\,[(p+q)^j_{i-1/2} v^{j-1/2}_i - (p+q)^j_{i+1/2} v^{j-1/2}_{i+1}], \\
p\theta = (\gamma - 1)\,E.
\end{gathered} \tag{8}$$

We shall not use fractional subscripts and indices below, but shall agree to let v^j_i refer to the point $(x_i,\ t_{j-1/2})$, p^j_i, q^j_i, θ^j_i, E^j_i to the point $(x_{i+1/2},\ t_j)$ where $t_{j-1/2} = \tau \times (j - 1/2)$, $x_{i+1/2} = h(i + 1/2)$.

2. Let us solve the problem of the motion of a stationary shock wave, which was considered in I, using the "cross" scheme. The progressive wave $u = f(x \pm Dt)$ satisfies the equation $Du_x \pm u_t$.

By analogy with this we define the difference progressive waves using the relations

$$u^{j+1}_i - u^j_i = \pm D\gamma_0\,(u^j_i - u^j_{i-1})$$

or

$$u^{j+1}_i - u^j_i = \pm D\gamma_0\,(u^j_{i+1} - u^j_i).$$

We shall look for the solution of the difference equations (8) in the form of progressive waves, writing

$$v_i^{j+1} - v_i^j = -\gamma_0 D (v_{i+1}^j - v_i^j), \quad \theta_i^{j+1} - \theta_i^j = -\gamma_0 D (\theta_i^j - \theta_{i-1}^j),$$
$$w_i^{j+1} - w_i^j = \gamma_0 D (w_i^j - w_{i-1}^j), \tag{9}$$

where

$$w_i^j = E_i^j + 0.5 (v_{i+1}^j)^2.$$

Using (8) and (9) we find

$$D \cdot v_{i+1} = P_i, \quad v_i = D(\overline{\theta}_1 - \theta_{i-1}), \quad Dw_i = P_i v_{i+1} = Dv_{i+1}^2 \tag{10}$$

(or $E_i = 0.5v_{i+1}^2$). The index is the same everywhere here, and so it has been omitted. Also $P_i = q_i = p_i$.

Using the Hugoniot condition, we find from (10)

$$q_i \theta_i = 0.5 (\gamma + 1) D^2 (\overline{\theta_1} - \theta_i)(\theta_i - \overline{\theta}_2). \tag{11}$$

From formula (3) with $\kappa = 1$ and $\nu = \nu_0 h^{1+\mu}$ we obtain

$$q_i \theta_i = 0.5\nu_0 \, | v_{i+1} - v_i |^{\mu} \, [| v_{i+1} - v_i | - (v_{i+1} - v_i)]. \tag{12}$$

From (11) and (12) we obtain an equation for θ_i:

$$0.5 (\gamma + 1) D^{1-\mu} (\overline{\theta}_1 - \theta_i)(\theta_i - \overline{\theta}_2) = \nu_0 (\theta_i - \theta_{i-1})^{1+\mu}. \tag{13}$$

We introduce the new unknown η_i from the formula $\theta_i - \overline{\theta}_2 = \Delta\theta \times \eta_i$ ($\eta_i > 0$) and obtain for it the equation

$$\eta_{i-1} = \eta_i - a\eta_i^{\sigma} (1 - \eta_i), \tag{14}$$

where $a = (1/\nu_0) \, [0.5 \times (\gamma + 1) \, (D \times \Delta\theta)^{1-\mu}]^{\sigma}$. Clearly $\eta_{\infty} = 1$, $\eta_{-\infty} = 0$.

3. Let us consider a linear viscosity ($\mu = 0$, $\sigma = 1$). In this case equation (14) takes the form

$$\eta_{i-1} = \eta_i - a\eta_i (1 - \eta_i). \tag{15}$$

We shall look for a monotonic non-negative solution of this equation. It exists for any $0 < a < 1$. From the condition $a < 1$ we find

$$\nu_0 \geqslant \nu_{0,\,cr} = 0.5 (\gamma + 1) D \cdot \Delta\theta. \tag{16}$$

For Neumann's viscosity ($\mu = 1$, $\sigma = 0.5$) we have

$$\eta_{i-1} = \eta_i - a \sqrt{\eta_i} (1 - \eta_i). \tag{17}$$

Let $\eta_i \geqslant 1 - \epsilon$ where ϵ is given, $0 < \epsilon < 0.5$. The least value of n for which $\eta_{i-n} \geqslant \epsilon$ is the effective width of the difference shock layer. L_ϵ Given ϵ it is easy to calculate L_ϵ for any a from formulae (15) and (17). The values of L_ϵ for $\epsilon = 0.1$ and $\epsilon = 0.05$ are given in the table.

Using a table such as this, it is easy to demonstrate the rules for selecting the coefficient of viscosity ν_0. Thus, we can find a_n for given ϵ and $L_\epsilon = n$ from the table. Clearly for all $a > a_n$ we obtain the effective width L_ϵ not exceeding the given value of n. Using this condition and the inequality (16) we find for a linear viscosity

$$\frac{\gamma+1}{2} \cdot D \cdot \Delta\theta \leqslant \nu_0 \leqslant \frac{\gamma+1}{2} \cdot D \cdot \Delta\theta \cdot \frac{1}{a_n}. \tag{18}$$

For the Neumann viscosity we find

$$\frac{\gamma+1}{2(1-\varepsilon)} \varepsilon^2 \leqslant \nu_0 \leqslant \frac{\gamma+1}{2a_n^2}. \tag{19}$$

The left-hand inequality of (19) is obtained from the requirement that η_{n-1} is non-negative, if $\eta_n < 1 - \epsilon$. We note that then the width of the

μ	ε	a_n								
		0.2	0.4	0.6	0.8	1.0	1.5	2.0	3.0	4.0
0	0.1	22	11	8	6	5	—	—	—	—
	0.05	29	15	9	7	6	—	—	—	—
1	0.1	16	9	7	5	4	3	3	2	2
	0.05	21	11	8	6	5	4	3	3	2

difference shock layer is infinite, although for the differential equation the width of the shock layer is finite. With the same plan we have looked at viscosity terms q of the form

$$q\theta = -0{,}5\,|v|\,(v_x - |v_x|)\,\nu \text{ and } q\theta = -0{,}5\nu\,(|v| + \beta_0)(v_x - |v_x|),$$
$$\beta_0 > 0 \text{ (small)}.$$

4. We used the "cross" difference scheme with viscosity to compute the motion of a stationary shock wave, the decay of a discontinuity, and other cases. These problems were calculated with a linear viscosity and with a Neumann viscosity. It always proved to be possible to use 3 to 4 times as large a time step with a linear viscosity as with a Neumann viscosity. However, with a Neumann viscosity it was possible to have a smaller effective width of the shock layer. In a number of cases the numerical

solution was compared with the exact solution of the problem, and each time the computation with a linear viscosity gave very good accuracy.

REFERENCES

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