We consider finite-difference schemes with "through computation" (i.e., which do not exclude the lines of discontinuity) for the equations of gas dynamics for one-dimensional isentropic motion of a gas with various types of viscosity. The progressive difference wave is defined. It is established that the difference equations can be solved in the form of such a wave. The requirement that the profile of the progressive wave shall be monotonic enables us to obtain a condition for the choice of the viscosity coefficient.

1. The equations of the one-dimensional isentropic motion of a gas in Lagrange variables have the form

\[ v_t + (p + q)_x = 0, \quad \theta_t = v_x, \quad (E + 0.5 v^2)_t + [(p + q)^{\theta}]_x = 0, \quad p\theta = (\gamma - 1) E, \]

where \( q \) is the viscosity \([1]\), \( p \) is the pressure, \( v \) the velocity, \( \theta \) the specific volume, \( E \) the internal energy, \( f_x, f_t \) the partial derivatives with respect to \( x \) and \( t \), and \( \gamma = c_p/c_v \).

Let us consider the problem of the motion of a stationary shock wave, which is spreading with a constant velocity \( D \). Then \( p(+\infty) = p_1 = 0; \)
\( p(-\infty) = p_2, \theta(+\infty) = \theta_1, \theta(-\infty) = \theta_2, v(-\infty) = v_2, v(+\infty) = v_1 = 0. \) We shall look for the solution as a function of \( s, f = f(s) \), where \( s = x - Dt \).
The problem reduces to the solution of a system of ordinary differential equations. We can easily find the integral of this system

\[ q \theta = 0.5 (\gamma + 1) D^2 (\theta_1 - \theta)(\theta - \theta_2), \quad q (| \infty |) = 0. \tag{2} \]

The requirements on \( q(s) \) are: 1) the system of ordinary differential equations to which our problem has led must have a continuous solution; 2) the effect of \( q(s) \) must be negligibly small outside the shock layer and in the region of the rarefied wave; 3) when the dimensions of the region of motion are large in comparison with the thickness of the shock layer, Hugoniot's conditions must be satisfied.

Generally speaking, \( q \) can be a function of \( \nu, \rho, \theta, E \) and their derivatives. We consider here the following expression for \( q \):

\[ \theta q = -0.5 \nu | v_x | ^{1+\mu} (v_x - \kappa | v_x |), \quad v = \text{const}. \tag{3} \]

When \( \mu = 1, \kappa = 0 \) we have Neumann's viscosity [1], which does not satisfy the second of the requirements. When \( \mu = 1, \kappa = 1 \) we have a viscosity which satisfies the second of the requirements (see [2]). When \( \mu = 0 \) we have a linear viscosity. We shall assume that \( 0 < \mu < 1 \).

2. Let us find the spread of the front of the shock wave caused by the viscosity. In the zone of the shock wave \( v_x < 0 \), and so \( \theta q = \nu \nu_x ^{1+\mu}, \quad \kappa = 1 \). For our problem we have \( \theta q = \nu D^{1+\mu}(\theta')^{1+\mu} \). We introduce the new function \( \lambda(s) \) with the formula \( \theta = 0.5(\theta_1 + \theta_2) + 0.5 \times \Delta \theta \times \lambda(s) \) where \( \Delta \theta = \theta_1 - \theta_2 \). If \( \theta = \theta_1 \) then \( \lambda(s) = 1 \), if \( \theta = \theta_2 \) then \( \lambda(s) = -1 \).

Using (2) we easily find

\[ \lambda'(s) = a [1 - \lambda^2(s)]^\sigma, \quad \sigma = \frac{1}{1+\mu}, \quad a = \left[ 0.5 \frac{\gamma + 1}{\nu} (0.5. D. \Delta \theta)^{1-\mu} \right]^\sigma. \tag{4} \]

Let \( s_2 \) denote the smallest \( s \) for which \( \lambda(s) = 1 \), and let \( s_1 \) denote the largest \( s \) for which \( \lambda(s) = -1 \). We shall call \( L = s_2 - s_1 \) the width of the shock layer. Integrating (4) we find

\[ I = aL, \quad \text{where} \quad I = \int_{-1}^{1} \frac{d\lambda}{(1 - \lambda^2)^{\sigma}} = V_\pi \frac{\Gamma(1-\sigma)}{\Gamma\left(\frac{3}{2} - \sigma\right)}. \]

When \( \mu = 1 (\sigma = 0.5) \) we have \( L = 2\nu/(\gamma + \nu)^{0.5\pi} \). If we put \( \nu = \nu_0 \times h^{1+\mu} \) and \( L = n \times h \) where \( n \) is an integer, we obtain

\[ \nu_0 = 0.5 (\gamma + 1) (0.5. D. \Delta \theta)^{1-\mu} h^{1+\mu} I^{1-\mu}. \tag{5} \]

From this formula we can see that \( \nu_0 \) is independent of the force of the shock wave only when \( \mu = 1 \). In this case we have
If $0 < \mu < 1$, then $L$ is finite. Further, if $\mu = 0$, then $L = \infty$. In this case it is convenient to introduce the concept of the effective width of the shock \label{1.11} 

3. Thus, let $\mu = 0$. By the effective width $L_\epsilon$ of the shock layer we shall mean the difference $L_\epsilon = s_2 - s_1$ where $s_1$ is the largest $s$ for which $\theta(s) - \overline{\theta}_1 = 0.5 \times \Delta \theta \times [1 + \lambda(s)] = \epsilon \times \Delta \theta$; $s_2$ is the smallest $s$ for which $\theta(s) - \overline{\theta}_1 = -0.5 \times \Delta \theta [1 - \lambda(s)] = -\epsilon \Delta \theta$ where $\epsilon$ is given, $0 < \epsilon < 1$. Then $dL_\epsilon = I_\epsilon = \int (1 + \lambda^2)^{-1} \Delta \theta$ where $a = 0.15/\nu(y + 1)D \Delta \theta$. Writing $L_\epsilon = n \times h$ and $\nu = \nu_0 h$ we find

\begin{equation}
\nu_0 \approx 0.25 \cdot (y + 1) \cdot D \cdot \Delta \theta \cdot n \cdot \ln^{-1} \left( \frac{1}{\epsilon} \right).
\end{equation}

II.

1. To write down the difference equations of gas dynamics we select a rectangular space-time net with steps $\Delta x = h$ and $\Delta t = r$. We shall use the "cross" difference scheme, which can be written in the form

\begin{equation}
\begin{align*}
\rho_i^{j+1/2} - \rho_i^{j-1/2} &= \gamma_0 [(p + q)_{i-1/2} - (p + q)_{i+1/2}], \quad \gamma_0 = \frac{r}{h}, \\
\theta_i^{j+1/2} - \theta_i^{j+1/2} &= \gamma_0 (v_i^{j+1/2} - v_i^{j-1/2}) \\
(\text{or} \quad h \cdot \theta_i^{j+1/2} &= r_i^{j+1/2} - r_i^{j-1/2}, \quad r_i^{j+1} = r_i^j + \tau v_i^{j+1/2}), \\
\rho_i^{j+1/2} - \rho_i^{j-1/2} &= 0.5 \left[ (\rho_i^{j+1/2})^2 - (\rho_i^{j-1/2})^2 \right] + \gamma_0 [(p + q)_{i-1/2} v_i^{j+1/2} - (p + q)_{i+1/2} v_i^{j+1/2}], \quad r_i^j = \frac{(r - 1) E}{h}.
\end{align*}
\end{equation}

We shall not use fractional subscripts and indices below, but shall agree to let $v_i^j$ refer to the point $(x_i, t_{i-j - 1/2}), \rho_i^j, q_i^j, \theta_i^j, E_i^j$ to the point $(x_i + 1/2, t_j)$. We let $t_{j - 1/2} = r \cdot (j - 1/2), x_i + 1/2 = h(i + 1/2).$

2. Let us solve the problem of the motion of a stationary shock wave, which was considered in I, using the "cross" scheme. The progressive wave $u = f(x \pm Dt)$ satisfies the equation $Du_x = u_t^*.$

By analogy with this we define the difference progressive waves using the relations

\begin{equation}
u_i^{j+1} - u_i^j = \pm D \gamma_0 (u_i^{j+1} - u_i^j)
\end{equation}
or

\begin{equation}
u_i^{j+1} - u_i^j = \pm D \gamma_0 (u_i^{j+1} - u_i^j)
\end{equation}
We shall look for the solution of the difference equations (8) in the form of progressive waves, writing

\[ v_{i+1}^j - v_i^j = -\gamma_0 D (v_{i+1}^j - v_i^j), \quad \theta_{i+1}^j - \theta_i^j = -\gamma_0 D (\theta_{i+1}^j - \theta_i^j), \]

\[ w_{i+1}^j - w_i^j = \gamma_0 D (w_{i+1}^j - w_i^j), \]

where

\[ w_i^j = \frac{4}{3} + 0.5 (v_{i+1}^j)^2. \]

Using (8) and (9) we find

\[ D \cdot v_{i+1} = P_i, \quad v_i = D (\theta_i - \theta_{i-1}), \quad Dw_i = P_i v_{i+1} = Dv_{i+1}^2 \]

(or \( E_i = 0.5v_{i+1}^2 \)). The index is the same everywhere here, and so it has been omitted. Also \( P_i = q_i = p_i \).

Using the Hugoniot condition, we find from (10)

\[ q_i \theta_i = 0.5 (\gamma + 1) \frac{\nu_0}{(\theta_i - \theta_{i-1}) (\theta_i - \theta_0)}. \]

From formula (3) with \( \kappa = 1 \) and \( \nu = \nu_0 \rho^{1+\mu} \) we obtain

\[ q_i \theta_i = 0.5 \nu_0 \left| v_{i+1} - v_i \right|^p \left| (v_{i+1} - v_i) - (v_{i+1} - v_i) \right|. \]

From (11) and (12) we obtain an equation for \( \theta_i \):

\[ 0.5 (\gamma + 1) D^{1-p} (\theta_i - \theta_{i-1}) (\theta_i - \theta_0) = \nu_0 (\theta_i - \theta_{i-1})^{1+p}. \]

We introduce the new unknown \( \eta_i \) from the formula \( \theta_i - \theta_2 = \Delta \theta \times \eta_i \) \((\eta_i > 0)\) and obtain for it the equation

\[ \eta_{i-1} = \eta_i - a \eta_i^p (1 - \eta_i), \]

where \( a = (1/\nu_0) \left[ 0.5 \times (\gamma + 1) (D \times \Delta \theta)^{1-p} \right] \). Clearly \( \eta_{\infty} = 1, \eta_{-\infty} = 0 \).

3. Let us consider a linear viscosity \((\mu = 0, \sigma = 1)\). In this case equation (14) takes the form

\[ \eta_{i-1} = \eta_i - a \eta_i^p (1 - \eta_i). \]

We shall look for a monotonic non-negative solution of this equation. It exists for any \( 0 < a < 1 \). From the condition \( a < 1 \) we find

\[ \nu_0 > \nu_{0,cr} = 0.5 (\gamma + 1) D \cdot \Delta \theta. \]

For Neumann's viscosity \((\mu = 1, \sigma = 0.5)\) we have

\[ \eta_{i-1} = \eta_i - a \eta_i (1 - \eta_i). \]
Let $\eta_{n-1} \geq 1 - \epsilon$ where $\epsilon$ is given, $0 < \epsilon < 0.5$. The least value of $n$ for which $\eta_{n-1} > \epsilon$ is the effective width of the difference shock layer. $L_{\varepsilon}$

Given $\epsilon$ it is easy to calculate $L_{\varepsilon}$ for any $a$ from formulae (15) and (17). The values of $L_{\varepsilon}$ for $\epsilon = 0.1$ and $\epsilon = 0.05$ are given in the table.

Using a table such as this, it is easy to demonstrate the rules for selecting the coefficient of viscosity $\nu_0$. Thus, we can find $a_n$ for given $\epsilon$ and $L_{\varepsilon} = n$ from the table. Clearly for all $a \geq a_n$, we obtain the effective width $L_{\varepsilon}$ not exceeding the given value of $n$. Using this condition and the inequality (16) we find for a linear viscosity

$$\frac{\gamma + 1}{2} \cdot D \cdot \Delta \theta \leq \nu_0 \leq \frac{\gamma + 1}{2} \cdot D \cdot \Delta \theta \cdot \frac{1}{a_n}. \quad (18)$$

For the Neumann viscosity we find

$$\frac{\gamma + 1}{2(1-\epsilon)} \leq \nu_0 \leq \frac{\gamma + 1}{2a_n}. \quad (19)$$

The left-hand inequality of (19) is obtained from the requirement that $\eta_{n-1}$ is non-negative, if $\eta_n < 1 - \epsilon$. We note that then the width of the difference shock layer is infinite, although for the differential equation the width of the shock layer is finite. With the same plan we have looked at viscosity terms $\eta$ of the form

$$q\theta = -0.5 |v| (|v_x| - |v_{\mu}|) \text{and } q\theta = -0.5 |v| (|v| + \beta_0 (|v_x| - |v_{\mu}|)).$$

$\beta_0 > 0$ (small).

4. We used the "cross" difference scheme with viscosity to compute the motion of a stationary shock wave, the decay of a discontinuity, and other cases. These problems were calculated with a linear viscosity and with a Neumann viscosity. It always proved to be possible to use 3 to 4 times as large a time step with a linear viscosity as with a Neumann viscosity. However, with a Neumann viscosity it was possible to have a smaller effective width of the shock layer. In a number of cases the numerical
solution was compared with the exact solution of the problem, and each time the computation with a linear viscosity gave very good accuracy.

REFERENCES


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