A PRIORI ESTIMATES FOR DIFFERENCE EQUATIONS*

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The present study arose in connection with an investigation of the convergence and accuracy of homogeneous difference through-computation schemes (see [1]) for the solution of non-linear and quasi-linear equations of the parabolic and hyperbolic types:

$$\mathcal{L}_1 u = \frac{\partial}{\partial x}\left[k(x,t,u)\frac{\partial u}{\partial x}\right] - f\left(x,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right) = 0, \tag{1}$$

$$\mathcal{L}_2 u = \frac{\partial}{\partial x}\left[k(x,t)\frac{\partial u}{\partial x}\right] - c(x,t)\frac{\partial^2 u}{\partial t^2} - f\left(x,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right) = 0, \tag{2}$$

$$\mathcal{L}_3 u = \frac{\partial^2}{\partial x^2}\left[k(x,t)\frac{\partial^2 u}{\partial x^2}\right] + c(x,t)\frac{\partial^2 u}{\partial t^2} + f\left(x,t,u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\frac{\partial u}{\partial t}\right) = 0 \tag{3}$$

in a bounded domain $\bar{D} = (0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant T)$ in the case when the coefficients $k(x,t,u)$ and $k(x,t)$ are discontinuous functions of (x,t) (see [2]).

Let $u = u(x,t)$ be a solution of the equation $\mathcal{L}_s u = 0$ $(s = 1,2,3)$ with certain initial data at $t = 0$ and boundary conditions (generally speaking, non-linear) for $x = 0$, $x = 1$, whilst y is a solution of the corresponding difference problem. The function $z = y - u$ is a measure of the accuracy of the difference method. It satisfies (given corresponding assumptions regarding f and $k(x,t,u)$) a non-homogeneous linear difference equation $\mathcal{P}_s z = \psi (s = 1,2,3)$, the right-hand side of which is the error of the approximation of the difference scheme in the solution of the differential equation $\mathcal{L}_s u = 0$. The boundary conditions of the 1st kind ($u(0,t) = u_1(t)$, $u(1,t) = u_2(t)$) are approximated accurately on the difference net. But if we take boundary conditions of the 2nd or 3rd kind or even of a more general type, as for instance (in the case of equation (1))

$$k\frac{\partial u}{\partial x} = C_1\frac{\partial u}{\partial x} + \sigma_1 u - u_1(t) \qquad \text{for } x = 1, \tag{4}$$

$$-k\frac{\partial u}{\partial x} = C_2\frac{\partial u}{\partial t} + \sigma_2 u - u_2(t) \qquad \text{for } x = 0, \tag{5}$$

a difference analogue of these conditions is obtained for y of the same order of approximation as the difference scheme (see [3]). It must be mentioned here that simple substitution in (4) and (5) of the derivatives by difference quotients yields difference boundary conditions of the first order of approximation.

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As a result we obtain for the function z non-homogeneous boundary conditions, analogous in form to (4) and (5), with right-hand sides ν_1 and ν_2.

The proof of convergence requires a uniform estimation of z in terms of ψ, ν_1 and ν_2 (*a priori* estimate).

In the case of discontinuous coefficients the position is complicated by the fact that, as the steps of the net tend to zero, ψ does not in general tend to zero close to points of discontinuity of the coefficient k. Hence estimation of ψ, maximally or in the mean, is insufficient for a proof of convergence. It had already been necessary in this connection to introduce the special norm $\|\psi\|_3$ when investigating the convergence of homogeneous schemes (see [1]) for the elementary stationary equation of heat conduction:

$$L^{(k,q,f)}u = \frac{d}{dx}\left[k(x)\frac{du}{dx}\right]-q(x)u+f(x) = 0. \tag{6}$$

We obtained in [4] *a priori* estimates, suitable for proving the convergence of difference schemes (for equation (1)) both in the case of fixed discontinuities of $k(x, t)$ (for $x =$ const.), and in the case of mobile discontinuities on a finite number of curves $x = \eta_m(t), m = 1, 2, \ldots, m_0$. These estimates were utilized in [5] to investigating the convergence and accuracy of straight-through computation schemes for the linear equation of heat conduction with boundary conditions of the 1st kind and a particular class of discontinuous coefficients.

Since the framework of one article does not permit a description to be given of the difference methods of solution of equations (1)–(3), and a statement and proof of the corresponding convergence theorems in the different classes of coefficients, we have been compelled to offer a separate treatment of the mathematical equipment enabling an estimate to be made of the error $z = y-u$. This is the aim of the present article. The *a priori* estimates obtained also allow the question of stability for given initial data and right-hand sides to be elucidated. It may be remarked that, for parabolic equations of the second order (and boundary conditions of the 1st kind), stability with respect to the right-hand side is sufficient for convergence. A separate treatment will be given of the theorems on convergence and accuracy of straight-through computation difference schemes of equations (1), (2) and (3) with boundary conditions of an extremely general type.

Our *a priori* estimates are obtained by using the method of integral ("energy") inequalities, which has become popular in the theory of differential equations, as also the method of distinguishing the "stationary" non-homogeneities, for which specially accurate estimates are obtained, since Green's difference function is used in their formation. We pay special attention here to the choice of norm for the estimate of the right-hand side and to weakening of the requirements imposed on the equation coefficients.

Certain elementary inequalities (for instance (2.21)) have appeared in works by other authors (see [6], [7], [8]); however, they were obtained for difference schemes of a particular kind and for a much narrower class of coefficients and boundary conditions of the first kind, and this made them unsuitable for our purposes.

Let us summarize the main features of the article. The introduction covers special notation and some auxiliary relations (Green's difference formulae, inequalities etc.) used in what follows.

In § 1, which is in essence a continuation of [1], we consider difference boundary problems for the stationary equation of heat conduction (6), for systems of equations, and for the fourth order equation

$$\frac{d^2}{dx^2}\left[k(x)\frac{d^2u}{dx^2}\right]-\frac{d}{dx}\left[b(x)\frac{du}{dx}\right]+q(x)u=f(x)$$

in the case of discontinuous $k(x)$. The estimates obtained in § 1, are utilized in §§ 2, 3 and 4 to make the estimate more precise in the case of an awkward right-hand side ψ.

Notice that an analogue of Theorem 5 holds for the case of certain spatial variables. We do not quote here the corresponding estimates.

In § 2, we give more precision to the results of [4] and deduce new estimates. In particular, estimates are obtained for systems of equations of the parabolic type.

In § 3 a number of *a priori* estimates is obtained for nine- and seven-point difference schemes, corresponding to (2). The simplest of them (3.18) was obtained in [9] with the auxiliary condition $|a_{\bar{x}}| < M$ in the case smooth coefficients for the implicit scheme ($\alpha_1 = 1$) and boundary conditions of the 1st kind.

Problems are investigated in § 4 for the difference analogue of a fourth order equation of the parabolic type.

Estimates of the same type as for difference equations hold for differential-difference equations obtained by Rothe's method and the method of straight lines.

All the results of §§ 1, 2 and 3 were obtained for non-uniform nets. However, the proofs are given for uniform nets for the sake of simplifying the treatment.

The numbering of the formulae is individual to each section. Double numbering is used to refer to a formula from another section (for instance, (B. 22) means formula (22) of the Introduction, (2.7) is formula (7) of § 2, and so on).

INTRODUCTION

1. *The difference net and net functions.* We take a domain $\bar{D} = (0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant T)$. We write $\bar{\Omega}$ for the difference net in $\bar{D}$, i.e. the set of points (x_i, t_j), where $x_i = ih, i = 0, 1, 2, \ldots, N, h = 1/N;\ t_j = j\tau, j = 0, 1, 2, \ldots, L,\ \tau = T/L$.

Let Ω_1 be the set of interior points (x_i, t_j) of the net $\bar{\Omega}$, for which $1 \leqslant i \leqslant N-1$, $1 \leqslant j \leqslant L$, Ω_2 the set of points (x_i, t_j), where $1 \leqslant i \leqslant N-1, 2 \leqslant j \leqslant L$, Ω_3 the set of points (x_i, t_j), where $2 \leqslant i \leqslant N-2, 2 \leqslant j \leqslant L$.

We agree to write net functions without net indices, i.e. instead of z_i^j we simply write z or $z(x, t)$, whilst also omitting the dependence of z on the steps h and τ of the net.

We simplify the writing by introducing the notation (see [4]):

$$z^{(+1)} = z^j_{i+1},\ z^{(-1)} = z^j_{i-1},\ \check{z} = z^{j-1}_i,\ \check{\check{z}} = z^{j-2},$$

$$z_{\bar{t}} = \frac{z-\check{z}}{\tau},\ z_{\bar{x}} = \frac{z-z^{(-1)}}{h},\ z_x = \frac{z^{(+1)}-z}{h},\ z_{x*} = 0\cdot 5\,(z_x+z_{\bar{x}}).$$

We now have

$$\frac{1}{h^2}\,[a_{i+1}(z_{i+1}-z_i)-a_i(z_i-z_{i-1})] = (az_{\bar{x}})_x,$$

2. *Sums and norms.* We introduce the following notation for sums [4]:

$$(\varphi,\psi) = \sum_{i=1}^{N-1}\varphi_i\psi_i h,\ [\varphi,\psi) = \sum_{i=0}^{N-1}\varphi_i\psi_i h,\ (\varphi,\psi] = \sum_{i=1}^{N}\varphi_i\psi_i h,\ [\varphi,\psi] = \sum_{i=0}^{N}\varphi_i\psi_i h,$$

where φ_i and ψ_i are any net functions.

Use is made below of the norms:

$$\|\psi\|_0 = \max_{0\leqslant i\leqslant N}|\psi_i|,\qquad \|\psi\|_m = (|\psi|_m, 1)^{1/m},\qquad m = 1, 2, \tag{1}$$

$$\|\psi\|_3 = \|\eta\|_2,\qquad \eta(x) = \sum_{x'=h}^{x'=x} h\psi(x'),\qquad \|\psi\|_{3*} = \|\eta\|_1, \tag{2}$$

$$\|\psi\|_4 = \|\psi\|_3+|(\psi, 1)|,\qquad \|\psi\|_{4*} = \|\psi\|_{3*}+|(\psi, 1)|. \tag{3}$$

When estimating the solutions of difference equations with right-hand side ψ and non-homogeneous boundary conditions, use is made of the norms

$$\|\psi\|_5 = \|\psi\|_4+|\nu_1|+|\nu_2|,\qquad \|\psi\|_{5*} = \|\psi\|_{4*}+|\nu_1|+|\nu_2|, \tag{4}$$

where $\nu_1 = \nu_1(t)$, and $\nu_2 = \nu_2(t)$ are the right-hand sides of the boundary conditions when $x = 0$ and $x = 1$.

Moreover, we shall write

$$\|y_{\bar{x}}\|_2 = (y^2_{\bar{x}}, 1]^{\frac{1}{2}} = [y^2_x, 1)^{\frac{1}{2}} = \|y_x\|_2\qquad (y^2_{\bar{x}} = (y_{\bar{x}})^2). \tag{5}$$

When investigating (in §§ 1 and 4) the fourth order difference operator $(ay_{\bar{x}x})_{\bar{x}x}$, we also encounter the sums

$$((\varphi,\psi)) = \sum_{i=2}^{N-2}\varphi_i\psi_i h,\qquad (\varphi,\psi)) = \sum_{i=1}^{N-2}\varphi_i\psi_i h,\qquad ((\varphi,\psi) = \sum_{i=2}^{N-1}\varphi_i\psi_i h. \tag{6}$$

If ψ is a function defined on the net $\bar{\Omega}$ or on part of it $\Omega_s (s = 1, 2, 3)$, the norm $\|\psi\|_m$ is a net function depending only on t and defined on the net $\omega_\tau = \{t_j, j = 0, 1, 2, \ldots, L\}$ or on part of it. We shall write

$$\|\widetilde{\psi}\|_m = \max_{(t)}\|\psi(x,t)\|_m,\qquad m = 1, 2, 3, 4, 5, 3^*, 4^*, 5^*. \tag{7}$$

Sums often encountered later are

$$\sum_{j'=s}^{j}\tau(\|\psi\|^{j'}_m)^2 = \sum_{j'=s}^{j}\tau\|\psi^{j'}\|^2_m = \sum_{t'=s\tau}^{t'=t}\tau\|\psi(x,t')\|^2_m,\qquad s = 0, 1, 2. \tag{8}$$

If $\psi = (\psi^k(x, t))$ is a net vector function, then

$$\|\psi\|_0 = \max_{(x)} \sqrt{\sum_k (\psi^k)^2}, \qquad \|\psi\|_m = \sum_k (|\psi^k|^m, 1)^{\frac{1}{m}}, \qquad m = 1, 2, \tag{9}$$

and so on.

3. *Non-uniform nets.* Suppose now that the net $\bar{\Omega} = \{x_i, t_i, 0 < j \leqslant N, 0 \leqslant j \leqslant L\}$ is non-uniform, i.e. its steps $h_i = x_i - x_{i-1}$ and $\tau_j = t_j - t_{j-1}$ vary from one point to the next.

We shall assume throughout that the steps h_i of the net $\omega_h = \{x_i, 0 \leqslant i \leqslant N\}$ satisfy the condition

$$0 < \varkappa_0 \leqslant \frac{h_{i+1}}{h_i} \leqslant \varkappa_1, \tag{10}$$

where $\varkappa_0$ and $\varkappa_1$ are constants, independent both of i and N.

A system of notation without indices is also used on a non-uniform net.

We shall write

$$z_{\bar{x}} = \frac{z_i^j - z_{i-1}^j}{h_i} = \frac{z - z^{(-1)}}{h}, \quad z_x = z_{\bar{x}}^{(+1)}, \quad z_{\bar{t}} = \frac{z_i^j - z_i^{j-1}}{\tau_j} = \frac{z - \check{z}}{\tau}.$$

In addition, the difference ratios are brought in:

$$z_{\tilde{x}} = \frac{z^{(+1)} - z}{\hbar}, \qquad \check{z}_{\tilde{t}} = \frac{z - \check{z}}{\bar{\tau}},$$

where

$$\hbar = \hbar_i = 0{\cdot}5(h_i + h_{i+1}), \qquad \bar{\tau}_j = 0{\cdot}5(\tau_j + \tau_{j-1}).$$

It should be borne in mind here that $z_{\bar{x}\tilde{x}} \neq z_{\tilde{x}\bar{x}}$.

The difference operator of the form $(az_{\bar{x}})_x$ on a uniform net is replaced on a non-uniform net by the operator $(az_{\bar{x}})_{\tilde{x}}$, whilst the operator $\check{z}_{\bar{t}\tilde{t}}$ is taken instead of $z_{\bar{t}t}$. In addition to the sums

$$(\varphi, \psi) = \sum_{i=1}^{N} \psi_i \varphi_i h_i, \qquad (\varphi, \psi] = \sum_{i=1}^{N} \varphi_i \psi_i h_i \qquad \text{etc.} \tag{11}$$

use is made of the sums

$$(\varphi, \psi)^* = \sum_{i=1}^{N-1} \varphi_i \psi_i \hbar_i \qquad \text{etc.} \tag{12}$$

In this case, as well as the norms $\|\psi\|_m$ $(m = 1, 2, 3)$, defined by analogy with (1), we shall make use of the norms

$$\|\psi\|_m^* = \{(|\psi|^m, 1)^*\}^{\frac{1}{m}}, \qquad m = 1, 2, \tag{13}$$

$$\|\psi\|_3^* = \|\eta\|_2^*, \qquad \eta(x) = \sum_{x'=h}^{x'=x} \psi(x') \hbar(x'). \tag{14}$$

We define similarly $\|\psi\|_m^*$, $m = 3^*$, 4, 5, 5*.

4. *Some elementary inequalities.* The mathematical apparatus that we use below consists of Green's difference formulae and certain elementary inequalities.

Let us enumerate first the following inequalities [11]:

1) The Cauchy-Bunyakovskii and Hölder inequalities:

$$|(y,z)| \leqslant \|y\|_2 \|z\|_2,$$

$$|(y,z)| \leqslant (|y|^p, 1)^{1/p} (|y|^q, 1)^{1/q} \qquad 1/p+1/q=1, p>0, q>0;$$

2)
$$|yz| \leqslant y^2/4c + cz^2, \tag{15}$$

where c is an arbitrary positive constant;

3)
$$\Big(\sum_{s=1}^{n} m_s\Big)^2 \leqslant n \sum_{s=1}^{n} m_s^2; \tag{16}$$

4)
$$\prod_{s=1}^{n} m_s^{\nu_s} \leqslant \sum_{s=1}^{n} \nu_s m_s, \qquad m_s \geqslant 0, \nu_s \geqslant 0, \sum_{s=1}^{n} \nu_s = 1. \tag{17}$$

Lemma 1. *Let $E(t)$ and $f(t)$ be two non-negative net functions, $E(t)$ being defined on the net $\{t_j = j\tau, j = s, s+1, \dots, L;\ s = 0, 1\}$, and $f(t)$ on the net $\{t_j = j\tau, j = s+1, s+2, \dots, L;\ s = 0, 1\}$. If*

$$E(t) \leqslant (1+M\tau)(E(t-\tau)+\tau f(t)), \tag{18}$$

then

$$E(t) \leqslant e^{Mt}\Big[E(s\tau) + \sum_{t'=(s+1)\tau}^{t'=t} \tau f(t')\Big]. \tag{19}$$

For we obtain, on successively applying inequality (18):

$$E(t) \leqslant q^{j-s}E(s\tau) + \sum_{k=0}^{j-s-1} q^{k+1} f(t-k\tau)\tau, \qquad q = 1+M\tau \qquad (t = j\tau).$$

Inequality (19) follows from this.

A similar inequality is obtained for the function $E(t)$, satisfying $E(t) \leqslant (1+M\tau)[E(t-2\tau)+\tau f(t)]$. In this case we introduce the function $E_1(t) = E(t)+E(t-\tau)$, for which inequality (18) holds.

5. *Green's difference formulae.* By using the obvious formulae for summation by parts:

$$(y, z_{\bar{x}}) = -[z, y_x) + y_N z_{N-1} - y_0 z_0 \qquad y_0 = y(0), y_N = y(1)), \tag{20}$$

$$(y, z_x) = -(z, y_{\bar{x}}] + y_N z_N - z_1 y_0, \tag{21}$$

we can readily obtain Green's formulae:

1) Green's first difference formula

$$(y, (az_{\bar{x}})_x) = -(a, y_{\bar{x}} z_{\bar{x}}] + (ayz_{\bar{x}})_N - a_1(y_x z)_0; \tag{22}$$

2) Green's second difference formula

$$(y, (az_{\bar{x}})_x) - (z, (ay_{\bar{x}})_x) = a_N(yz_{\bar{x}} - zy_{\bar{x}})_N - a_1(yz_x - y_x z)_0. \tag{23}$$

Green's formulae become on a non-uniform net:

$$(y, (az_{\bar{x}})_{\bar{x}})^* = -(a, y_{\bar{x}} z_{\bar{x}}] + (ayz_{\bar{x}})_N - a_1(y_x z)_0, \tag{24}$$

$$(y, (az_{\bar{x}})_{\bar{x}})^* - (z, (ay_{\bar{x}})_{\bar{x}})^* = a_N(yz_{\bar{x}} - zy_{\bar{x}})_N - a_1(yz_x - y_x z)_0. \tag{25}$$

We now consider the fourth order difference operator $(ay_{\bar{x}x})_{\bar{x}x}$ and show that the following Green's difference formula holds for it:

$$((y, (az_{\bar{x}x})_{\bar{x}x})) = (a, y_{\bar{x}x} z_{\bar{x}x}) + [y^{(-1)}(a^{(-1)} z_{\bar{x}\bar{x}})_{\bar{x}} - a^{(-1)} y_{\bar{x}} z_{\bar{x}\bar{x}}]_N - \\ -[y^{(+1)}(a^{(+1)} z_{xx})_x - a^{(+1)} y_x z_{xx}]_0. \tag{26}$$

We make use of Green's second difference formula for the operator $w_{\bar{x}x}$ in the domain $h \leqslant x \leqslant x_{N-1} = 1-h$:

$$((y, w_{\bar{x}x})) = ((y_{\bar{x}x}, w)) + (yw_{\bar{x}} - y_{\bar{x}} w)_{N-1} - (yw_x - y_x w)_1.$$

By taking into account the identity $y_{\bar{x}} = y_x - hy_{\bar{x}x}$, we transform this to the form

$$((y, w_{\bar{x}x})) = (y_{\bar{x}x}, w) + (y^{(-1)} w_{\bar{x}}^{(-1)} - y_{\bar{x}} w^{(-1)})_N - (y^{(+1)} w_x^{(+1)} - w^{(+1)} y_x)_0. \tag{26'}$$

Hence (26) follows, after substituting $w = az_{\bar{x}x}$.

On interchanging the roles of y and z in (26) and subtracting the resulting identity from (26), we get Green's second formula for the fourth order difference operator. We shall not write it down.

If, for instance, the following conditions are fulfilled:

$$y = y_x = z = z_x = 0 \quad \text{for } x = x_0 = 0; \qquad y = y_x = z = z_x = 0 \quad \text{for } x = x_N = 1$$

or

$$y_x = z_x = (a^{(+1)} y_{xx}) = (a^{(+1)} z_{xx})_x = 0 \qquad \text{for } x = 0,$$
$$y_{\bar{x}\bar{x}} = z_{\bar{x}\bar{x}} = (a^{(-1)} y_{\bar{x}\bar{x}})_{\bar{x}} = (a^{(-1)} z_{\bar{x}\bar{x}})_{\bar{x}} = 0 \qquad \text{for } x = 1;$$

all the substitutions vanish and Green's formulae become

$$((y, (az_{\bar{x}x})_{\bar{x}x})) = (a, y_{\bar{x}x} z_{\bar{x}x}), \qquad ((y, (az_{\bar{x}x})_{\bar{x}x})) = ((z, (ay_{\bar{x}x})_{\bar{x}x})). \tag{27}$$

In particular, we have with $y = z$:

$$((z, (az_{\bar{x}x})_{\bar{x}x})) = (a, z_{\bar{x}x}^2).$$

We shall conclude this section by mentioning one point in connection with the notation. All constants, independent of the net, will be denoted by the letter M; their structure and connection with the initial constants will often be ignored, since the connection can easily be ascertained from the exposition.

§ 1. STATIONARY PROBLEMS

An *a priori* estimate of the form

$$(aw_{\bar{x}})_x - dw = -\psi, \qquad w_0 = w_N = 0.$$

was obtained in [1] for the solution of the first boundary problem

$$\|w\|_0 \leqslant M \|\psi\|_3^*$$

Estimates for w, $w_{\bar{x}}$ and $w_{\bar{t}}$ are obtained by similar methods in this section (for the case when the coefficients of the equation depend on t) for the third boundary problem, together with estimates of the solutions of the fourth order equation $(aw_{\bar{x}x})_{\bar{x}x} = \psi$ and of a system of second order equations.

1. *Green's difference function.* We consider on the net $\omega_h = \{x_i = ih;\ 0 \leqslant i \leqslant N\}$ the difference boundary problem

$$L_h w = (aw_{\bar{x}})_x - dw = -\psi \qquad (x = x_i,\ 0 < i < N), \tag{1}$$

$$a_1 w_{x,0} = \sigma_1 w_0 - \nu_1 \qquad -a_N w_{\bar{x},N} = \sigma_2 w_N - \nu_2, \tag{2}$$

the coefficients of which satisfy the conditions

$$a \geqslant M_0 > 0,\ d \geqslant 0,\ \sigma_1 \geqslant 0,\ \sigma_2 \geqslant 0,\ \sigma_1 + \sigma_2 \geqslant M_1 > 0, \tag{3}$$

where M_0, M_1 are constants independent of h.

Our aim is to estimate $\|w\|_0$ and $\|w_{\bar{x}}\|_m (m = 1, 2)$. We do this by using Green's difference function $G(x, \xi)$ of problem (1)–(3) (as usual, we give no indication of the dependence of G on h). The function $G(x, \xi)$ is defined by the conditions

$$(a(x)G_{\bar{x}}(x,\xi))_x - d(x)G(x,\xi) = -\frac{\delta(x,\xi)}{h}, \qquad \delta(x,\xi) = \begin{cases} 0, & x = \xi, \\ 1, & x \neq \xi, \end{cases} \tag{4}$$

$$a(h)G_x(0,\xi) = \sigma_1 G(0,\xi), \qquad -a(1)G_{\bar{x}}(1,\xi) = \sigma_2 G(1,\xi) \tag{5}$$

and can be written in the form

$$G(x,\xi) = \frac{1}{\Delta}\alpha(x)\beta(\xi) \qquad \text{for } x < \xi, \qquad G(x,\xi) = \frac{1}{\Delta}\alpha(\xi)\beta(x) \qquad \text{for } x > \xi, \tag{6}$$

where

$$\Delta = \text{const.} \geqslant M > 0 \tag{7}$$

(see (12)), whilst $\alpha(x)$ and $\beta(x)$ are found from the conditions

$$\begin{aligned} &L_h\alpha = 0, \qquad a_1\alpha_{x,0} = 1, \qquad a_1\alpha_{x,0} = \sigma_1\alpha_0; \\ &L_h\beta = 0, \quad a_N\beta_{\bar{x},N} = -1, \quad -a_N\beta_{\bar{x},N} = \sigma_2\beta_N. \end{aligned} \tag{8}$$

If $d \equiv 0$, we have

$$\alpha(x) = \frac{1}{\sigma_1} + \sum_{x'=h}^{x'=x}\frac{h}{a(x')}, \qquad \beta(x) = \sum_{x'=x+h}^{x'=1}\frac{h}{a(x')} + \frac{1}{\sigma_2},$$
$$\Delta = \alpha(x) + \beta(x) = \text{const.} \tag{9}$$

If $\sigma_1 = 0$, we put $a_1\alpha_{x,0} = 0$, $\alpha(0) = 1$.

We can prove, by analogy with [1]:

Lemma 2. *If $G(x,\xi)$ is Green's function of problem* (1)–(3), *we have*

$$\|G\|_0 \leqslant M, \qquad \|G_{\bar{x}}\|_0 \leqslant M, \qquad \|G_{\bar{\xi}}\|_0 \leqslant M.$$

Lemma 3. *The following relations hold*:

$$|G_{\bar{x}\bar{\xi}}(x,\xi)| \leqslant M \qquad \text{for } x \neq \xi, \qquad h|G_{\bar{x}\bar{\xi}}(x,x)| \leqslant M.$$

We find, in fact, by making use of (6), that

$$G_{\bar{x}}(x, \xi) = \frac{1}{\Delta}\alpha_{\bar{x}}(x)\beta(\xi) \quad \text{for } x \leqslant \xi, \qquad G_{\bar{x}}(x, \xi) = \frac{1}{\Delta}\beta_{\bar{x}}(x)\alpha(\xi) \quad \text{for } x > \xi,$$

$$G_{\bar{x}\bar{\xi}}(x, \xi) = \frac{1}{\Delta}\alpha_{\bar{x}}(x)\beta_{\xi}(\xi) \qquad \text{for } x \leqslant \xi,$$

$$G_{\bar{x}\bar{\xi}}(x, \xi) = \frac{1}{\Delta}\beta_{\bar{x}}(x)\alpha_{\xi}(\xi) \qquad \text{for } x > \xi.$$

It follows from this, since $\alpha_{\bar{x}}$ and $\beta_{\bar{x}}$ are bounded, that $G_{\bar{x}\xi}$ is bounded. If, for instance, $d = 0$, then

$$G_{\bar{x}\bar{\xi}}(x, x) = -\frac{1}{ha(x)} + \alpha_{\bar{x}}(x)\beta_{\bar{x}}(x)/\Delta.$$

On putting $y = w$, $z = G$ in Green's second formula (B. 23), we obtain the following form for the solution of problem (1)–(3):

$$w(x) = (G(x, \xi), \psi(\xi)) + G(x\ 0)\nu_1 + G(x, 1)\nu_2. \tag{10}$$

2. *Estimates of* w *and* $w_{\bar{x}}$. We now make use of (10) and Lemmas 2 and 3 to estimate the solution of problem (1)–(3).

THEOREM 1. *If* w *is a solution of problem* (1)–(3), *we have*

$$\|w\|_0 \leqslant M\|\psi\|_{5*}, \qquad \|w_{\bar{x}}\|_1 \leqslant M\|\psi\|_{5*}, \tag{11}$$

$$|w_{x,0}| \leqslant M\|\psi\|_{5*}, \qquad |w_{\bar{x},N}| \leqslant M\|\psi\|_{5*}. \tag{12}$$

We introduce the function $\eta(x)$ with the supplementary conditions

$$\eta_{\bar{x}} = \psi, \qquad \eta_0 = 0, \qquad \eta(x) = \sum_{x'=h}^{x'=x} h\psi(x'). \tag{13}$$

Formula (B. 20) for summation by parts gives

$$(G, \psi) = (G(x, \xi), \eta_{\bar{\xi}}(\xi)) = -(G_{\xi}, \eta) + G(x, 1)\,\eta_{N-1}.$$

Hence it follows from (10), by virtue of Lemma 3, that $\|w\|_0 \leqslant M\|\psi\|_{5*}$. We now consider

$$w_{\bar{x}} = (G_{\bar{x}}, \psi) + G_{\bar{x}}(x, 0)\nu_1 + G_{\bar{x}}(x, 1)\nu_2. \tag{14}$$

On taking (13) into account, we find that

$$(G_{\bar{x}}, \psi) = -(G_{\bar{x}\xi}, \eta) + G_{\bar{x}}(x, 1)\eta_{N-1}. \tag{15}$$

On using the inequality

$$\|(G_{\bar{x}\xi}, \eta)\|_1 \leqslant \sum_{x'=h}^{x'=1}\{h(G_{\bar{x}\bar{\xi}}(x', \xi), \eta(\xi))' + h^2 G_{\bar{x}\bar{\xi}}(x', x')\eta(x'-h)\},$$

where the dashes indicate that the sum of the $(G_{\bar{x}\bar{\xi}}, \eta)'$ is taken for $\xi \neq x'-h$, together with Lemma 3, we find

$$\|(G_{\bar{x}\bar{\xi}}, \eta)\|_1 \leqslant M\|\eta\|_1 = M\|\psi\|_{3*}.$$

Hence it follows, from (14), (15), that $\|w_{\bar{x}}\|_1 \leqslant M\|\psi\|_{5*}$.

LEMMA 4. *If w is the solution of problem* (1)–(3), *we have*

$$\| w_{\bar{x}} \|_2 \leqslant M \| \psi \|_5. \tag{16}$$

The proof follows from (15) and Lemma 3. However, we shall use a different method of proof, characteristic of the discussion that follows and making no use of Green's function. We multiply (1) by wh, sum over the net $x = h, 2h, \ldots, (N-1)h$ and make use of (B. 22):

$$(a, w_{\bar{x}}^2] + (d, w^2) - a_N w_N w_{\bar{x}, N} + a_1 w_0 w_{x, 0} = (\psi, w)$$

or, if (2) is applied,

$$I + (d, w^2) = (\psi, w) + \nu_1 w_0 + \nu_2 w_N,$$

where

$$I = (a, w_{\bar{x}}^2] + \sigma_1 w_0^2 = \sigma_2 w_N^2.$$

We introduce a function $\eta(x)$ with the aid of conditions (13):

$$(w, \psi) = -(w_{\bar{x}}, \eta) + w_N \eta_{N-1}.$$

On using Bunyakovskii's inequality, together with the inequality

$$w^2 \leqslant MI \tag{16a}$$

(see [4]), Lemma 1*), we have

$$|(w_{\bar{x}}, \eta)| \leqslant \| w_{\bar{x}} \|_2 \| \eta \|_2 \leqslant M \sqrt{I} \| \psi \|_3, \qquad I + (d, w^2) \leqslant M \| \psi \|_5 \sqrt{I} \leqslant M \| \psi \|_5^2.$$

Inequality (16) follows from this, in view of (3).

We shall adduce an example, showing that $\|\psi\|_{4*}$ gives a more exact estimate for an awkward ψ function by comparison with $\|\psi\|_4$, not to mention $\|\psi\|_2$. We assume that

$$\nu_1 = \nu_2 \equiv 0, \qquad \psi = \psi_i = (\delta_{i, n+1} - \delta_{i, n}) \frac{1}{h^m} \qquad (m = 0, 1).$$

We now have:

$$\| \psi \|_2 = \frac{\sqrt{2}}{\sqrt{h}}, \quad \| \psi \|_4 = \| \psi \|_3 = \sqrt{h}, \quad \| \psi \|_{4*} = \| \psi \|_{3*} = h \qquad \text{for } m = 0,$$

$$\| \psi \|_2 = \sqrt{2} \sqrt{h}, \quad \| \psi \|_4 = h^{\frac{3}{2}}, \quad \| \psi \|_{4*} = h^2 \qquad \text{for } m = 1,$$

The advantage is clear from this of estimates obtained with the aid of Green's function, as compared with estimates obtained by the method of integral inequalities. This advantage tells in the case of poor ψ functions. Notice that the estimate $\|w\|_0 \leqslant \leqslant M \|\psi\|_5$ is obtained instead of $\|w\|_0 \leqslant M \|\psi\|_{5*}$ by the method of integral inequalities that we used in the proof of Lemma 4.

3. *Estimate of w_t.* Suppose now that the functions a, ψ, d, and hence w, depend on t and are defined on the net $\bar{\Omega}$, or more precisely, $a = a(x, t)$, $0 < x \leqslant 1$, $0 \leqslant t \leqslant T$, $\psi = \psi(x, t)$, $0 < x < 1$, $0 \leqslant t \leqslant T$, whilst the functions $\sigma_s(t)$, $\nu_s(t)$ $(s = 1, 2)$ are given on the net $\omega_\tau^T = \{t_j = j\tau, 0 \leqslant j \leqslant L\}$.

We also assume that a, σ_s, d satisfy Lipschitz conditions with respect to t, so that

$$|a_{\bar{t}}| \leqslant M, \qquad |d_{\bar{t}}| \leqslant M, \qquad |(\sigma_s)_{\bar{t}} \leqslant M \qquad s=1,2, \tag{17}$$

one of the following cases being realized:

$$\sigma_s \leqslant M \qquad \text{or} \quad \sigma_s = \infty. \tag{18}$$

If, for instance, $\sigma_1 = \infty$, we obtain with $x = 0$ the homogeneous boundary condition of the 1st kind $w_0 = 0$.

LEMMA 5. *If conditions* (1)–(3), (17) *and* (18) *are fulfilled, we have*

$$\| w_{\bar{t}} \|_0 \leqslant M (\| \psi \|_{5*} + \| \psi_{\bar{t}} \|_{5*}), \tag{19}$$

$$\| w_{x\bar{t}} \|_1 \leqslant M (\| \psi \|_{5*} + \| \psi_{\bar{t}} \|_{5*}), \tag{20}$$

where

$$\| \psi_{\bar{t}} \|_{5*} = \| \psi_{\bar{t}} \|_{4*} + | \nu_{1\bar{t}} | + | \nu_{2\bar{t}} |.$$

If, for instance, the condition $w_0 = 0$ is given at $x = 0$, we have to put formally $\nu_1(t) \equiv 0$ in (19) and (20).

On introducing the function $\zeta = w_{\bar{t}}$, we obtain for it the equations

$$(a\zeta_{\bar{x}})_x - d\zeta = -[\psi_{\bar{t}} + (a_{\bar{t}} \check{w}_{\bar{x}})_x - d_{\bar{t}} \check{w}] = -\bar{\psi}, \qquad a_1 \zeta_{x,0} = \sigma_1 \zeta_0 - \bar{\nu}_1,$$
$$-a_N \zeta_{\bar{x},N} = \sigma_2 \zeta_N - \nu_2,$$

where

$$\bar{\nu}_1 = \nu_{1\bar{t}} - \sigma_{1\bar{t}} \check{w}_0 + a_{\bar{t},1} \check{w}_{x,0}, \qquad \bar{\nu}_2 = \nu_{2\bar{t}} - \sigma_{2\bar{t}} \check{w}_N - a_{\bar{t},N} \check{w}_{\bar{x},N}.$$

The conditions of Theorem 1 are fulfilled, so that

$$\| \zeta \|_0 \leqslant M \| \bar{\psi} \|_{5*}, \qquad \| \bar{\psi} \|_{5*} = \| \bar{\psi} \|_{4*} + | \bar{\nu}_1 | + | \bar{\nu}_2 |.$$

By Theorem 1 and conditions (17):

$$| \bar{\nu}_s | \leqslant \nu_{s\bar{t}} | + M \| \psi \|_{5*}, \qquad s = 1,2.$$

On now observing that

$$\sum_{x'=h}^{x'=x} h(a_{\bar{t}} \check{w}_{\bar{x}})_x = a_{\bar{t}}^{(+1)} \check{w}_x - a_{\bar{t},1} \check{w}_{x,0},$$

and taking into account inequalities (12) of Theorem 1, we obtain (19).

Lemma 5 for the first boundary problem was utilized in [5] to prove the convergence of homogeneous difference schemes for the linear equation of heat conduction with discontinuous coefficients.

We can similarly prove

LEMMA 6. *If the conditions of Lemma* 5 *are satisfied, and in addition,*

$$|(\sigma_s)_{\bar{t}\bar{t}}| \leqslant M, \qquad |a_{\bar{t}\bar{t}}| \leqslant M, \qquad |d_{\bar{t}\bar{t}}| \leqslant M,$$

then

$$\| w_{\bar{t}\bar{t}} \|_0 \leqslant M \{ \| \psi \|_{5*} + \| \psi_{\bar{t}} \|_{5*} + \| \psi_{\bar{t}\bar{t}} \|_{5*} \}, \tag{21}$$

where

$$\| \psi_{\bar{t}\bar{t}} \|_{5*} = \| \psi_{\bar{t}\bar{t}4*} + | \nu_{1\bar{t}\bar{t}} | + | \nu_{2\bar{t}\bar{t}} |. \tag{22}$$

Account must be taken here of inequality (20).

All the results of Sections 1–3 retain their force for the solution of the boundary problem

$$(aw_{\bar{x}})_{\tilde{x}} - d \cdot w = -\psi \tag{23}$$

with conditions (2) and (3), defined on the non-uniform net

$$\omega_h = \{x_i, 0 \leqslant i \leqslant N\}.$$

Here, instead of $\|\psi\|_{5*}$ and $\|\psi\|_5$, we have to write in all the estimates the norms

$$\|\psi\|^*_{5*} \text{ and } \|\psi\|^*_5,$$

defined in accordance with formulae (B. 13) and (B. 14).

4. *System of second order equations.* We now consider the general case of a difference boundary problem for a system of the second order:

$$(\mathbf{a}\mathbf{w}_{\bar{x}x}) - \mathbf{d} \cdot \mathbf{w} = -\boldsymbol{\psi}, \tag{24}$$

$$\mathbf{w}(0) = 0, \qquad \mathbf{w}(1) = 0, \tag{25}$$

where $\mathbf{a} = (a_{ij})$ and $\mathbf{d} = (d_{ij})$ are matrices, $i, j = 1, 2, \ldots, r$; $\boldsymbol{\psi} = (\psi^j)$, $\mathbf{w} = (w^j)$ ($j = 1, 2, \ldots, r$) are vectors.†

We shall suppose that

$$\begin{aligned} &\sum_{i,j} a_{ij}\xi_i\xi_j \geqslant \beta \sum_j \xi_j^2, \qquad \beta > 0, \\ &\sum_{i,j} d_{ij}\xi_i\xi_j \geqslant \gamma \sum_j \xi_j^2, \qquad \gamma \geqslant 0, \end{aligned} \tag{26}$$

where $\{\xi_j\}$ are any real numbers, β and γ are real constants, not dependent on the net.

We shall confine ourselves for simplicity to the first boundary problem, although all the results will also hold for the boundary problem of the 3rd kind, analogous to problem (1)–(3) for a single equation.

We form the scalar product of (24) with the vector function $\mathbf{w}h$ and sum over the net from $x = h$ to $x = 1-h$. On making use of the formula for summation by parts (B. 20), we get Green's formula

$$\sum_{i,j} \{(a_{ij} \overset{j}{w}_{\bar{x}}, \overset{i}{w}_{\bar{x}}] + (d_{ij}w^j, w^i)\} = \sum_i (w^i, \psi^i), \tag{27}$$

or

$$(\mathbf{a}\mathbf{w}_{\bar{x}}, \mathbf{w}_{\bar{x}}] + (\mathbf{d}\mathbf{w}, \mathbf{w}) = \mathbf{w}, \boldsymbol{\psi}). \tag{27'}$$

On taking (26) into account, we have

$$\beta \|\mathbf{w}_{\bar{x}}\|_2^2 + \gamma \|\mathbf{w}\|_2^2 \leqslant (\mathbf{w}, \boldsymbol{\psi}). \tag{28}$$

On observing that $|\mathbf{w}| \leqslant \|\mathbf{w}_{\bar{x}}\|_2$, $|(\mathbf{w}, \boldsymbol{\psi})| \leqslant \|\mathbf{w}_{\bar{x}}\|_2 \|\boldsymbol{\psi}\|_2$, we get the first estimate:

$$\|\mathbf{w}\|_0 \leqslant \frac{1}{\beta} \|\boldsymbol{\psi}\|_2, \tag{29}$$

† The employment of the convenient indices i and j can hardly lead to misunderstanding, since we have agreed not to use them as net indices.

We now bring in the vector function $\boldsymbol{\eta} = \{\eta^j(x)\}$, $j = 1, 2, \ldots, r$, with the aid of the conditions $\boldsymbol{\eta}_x = \boldsymbol{\psi}$, $\boldsymbol{\eta}(0) = 0$, so that

$$\boldsymbol{\eta}(x) = \sum_{x'=h}^{x-h} h\boldsymbol{\psi}(x'),$$

and we use the formula for summation by parts (B. 21)

$$(\boldsymbol{\psi}, \mathbf{w}) = \sum_j (w^j, \psi^j) = (\mathbf{w}, \boldsymbol{\eta}_x) = -(\mathbf{w}_{\bar{x}}, \boldsymbol{\eta}]$$

and Bunyakovskii's inequality

$$|(\boldsymbol{\psi}, \mathbf{w})| \leqslant \|\mathbf{w}_{\bar{x}}\|_2 \|\boldsymbol{\psi}\|_3 \qquad (\|\boldsymbol{\eta}\|_2 = \|\boldsymbol{\psi}\|_3).$$

Returning to (28), we find that

$$\beta \|\mathbf{w}_{\bar{x}}\|_2^2 \leqslant \|\mathbf{w}_{\bar{x}}\|_2 \|\boldsymbol{\psi}\|_3 \quad \text{and} \quad \|\mathbf{w}_{\bar{x}}\|_2 \leqslant \frac{1}{\beta} \|\boldsymbol{\psi}\|_3.$$

We have now proved

THEOREM 2. *If* $\mathbf{w}$ *is the solution of problem* (24)–(26), *we have*

$$\|\mathbf{w}\|_0 \leqslant \frac{1}{\beta} \|\boldsymbol{\psi}\|_3, \tag{30}$$

$$\|\mathbf{w}_{\bar{x}}\|_2 \leqslant \frac{1}{\beta} \|\boldsymbol{\psi}\|_3. \tag{31}$$

The proof is similar for

LEMMA 7. *If the conditions*

$$|(a_{ij})_{\bar{t}}| \leqslant M, \quad |(d_{ij})_{\bar{t}}| \leqslant M \qquad (i = 1, 2, \ldots, r),$$

are satisfied then the following inequalities hold:

$$\|\mathbf{w}_{\bar{t}}\|_0 \leqslant M(\|\boldsymbol{\psi}\|_3 + \|\boldsymbol{\psi}_{\bar{t}}\|_3), \tag{32}$$

$$\|\mathbf{w}_{\bar{x}\bar{t}}\|_0 \leqslant M(\|\boldsymbol{\psi}\|_3 + \|\boldsymbol{\psi}_{\bar{t}}\|_3). \tag{33}$$

Lemma 7 is useful in forming *a priori* estimates for systems of non-stationary equations.

5. *Fourth order equations.* We consider the following boundary problem for a fourth order equation:

$$(aw_{\bar{x}x})_{\bar{x}x} - (bw_{\bar{x}})_x + d \cdot w = \psi \quad \text{on the net} \quad \{x_i = ih,\ 2 \leqslant i \leqslant N-2\}, \tag{34}$$

$$w_0 = w_{x,0} = 0, \quad w_N = w_{\bar{x},N} = 0, \tag{35}$$

$$a \geqslant M > 0, \quad b \geqslant 0, \quad d \geqslant 0. \tag{36}$$

THEOREM 3. *If* w *is a solution of problem* (34)–(36), *we have*

$$\|w\|_0 \leqslant M\|\psi\|_3, \quad \|w_{\bar{x}}\|_0 \leqslant M\|\psi\|_3 \quad \|w_{\bar{x}x}\|_2 \leqslant M\|\psi\|_3, \tag{37}$$

where $\|\psi\|_3 = ((\eta, \eta))^{\frac{1}{2}} = \|\eta\|_2$, *whilst* $\eta(x)$ *is the solution of the problem*

$$\eta_{\bar{x}x} = \psi, \quad \eta_1 = 0, \quad \eta_{x,1} = 0. \tag{38}$$

On putting $y = z = w$ in Green's formula (B. 26), we obtain

$$(a, w_{\bar{x}x}^2)+((b, w_{\bar{x}}^2)+((d, w^2)) = ((\psi, w)). \tag{39}$$

In view of conditions (35), formula (B. 26′) gives

$$((\psi, w)) = ((\eta_{\bar{x}x}, w)) = (\eta, w_{\bar{x}x})+w_{N-1}\eta_{\bar{x}, N-1}-\eta_{\bar{x}, N}w_{\bar{x}, N} = (\eta, w_{\bar{x}x}).$$

On next taking into account the relations (see [4])

$$\| w \|_0 \leqslant \| w_{\bar{x}} \|_0 \leqslant \| w_{\bar{x}x} \|_2, \quad | (\eta, w_{\bar{x}x}) | \leqslant \| \eta \|_2 \| w_{\bar{x}x} \|_2,$$

we obtain inequalities (37) from (38).

If the conditions $w_{\bar{x}x} = 0$, $(a^{(-1)} w_{\bar{x}x})_{\bar{x}x} = 0$ are given for $x = 1$, we have

$$\| w \|_0 \leqslant M \| \psi \|_4, \quad \| \psi \|_4 = \| \psi \|_3 + | ((\psi, 1)) |. \tag{40}$$

An analogue of Lemma 6 holds for the solution of problem (34)–(36).

It is not possible for us to dwell here on *a priori* estimates for solutions of equation (34) with boundary conditions of a higher order of accuracy.

§ 2. EQUATIONS OF THE PARABOLIC TYPE

1. *The initial problem.* We take the following problem in the domain $\bar{\Omega}$ for an equation which is the difference analogue of a linear differential equation of the parabolic type:

$$\mathcal{P}_1 z = \rho z_{\bar{t}} - (a z_{\bar{x}})_x^{(\alpha)} - Q(z) = \psi \qquad \text{on } \Omega_1, \tag{1}$$

$$\left.\begin{array}{lll} (a^{(+1)} z_x)^{(\alpha)} = \mathcal{E}_1(t) z_{\bar{t}} + (\sigma_1(t) z)^{(\alpha)} - \nu_1(t) & \text{for } x = 0, & t = \tau, 2\tau, \ldots, L\tau = T, \\ -(a z_{\bar{x}})^{(\alpha)} = \mathcal{E}_2(t) z_{\bar{t}} + (\sigma_2(t) z)^{(\alpha)} - \nu_2(t) & \text{for } x = 1, & t = \tau, 2\tau, \ldots, L\tau = T, \\ z^0 = z(x, 0) = \varphi(x), & \text{for } t = 0, & \end{array}\right\} \tag{2}$$

where the index α denotes summation over the rows t and $t-\tau$ of the net with weighting factors α_1 and α_2:

$$v^{(\alpha)} = \alpha_1 v + \alpha_2 \check{v}, \ \alpha_1 \geqslant 0, \ \alpha_2 \geqslant 0, \ \alpha_1 + \alpha_2 = 1,$$

$$Q(z) = (b_1 z^*)_x + (b_2 \check{z}^*)_x + \bar{b}_1 z_{x*} + \bar{b}_2 \check{z}_{x*} + d_1 z + d_2 \check{z}, \tag{3}$$

$$z^* = 0 \cdot 5(z + z^{(-1)}), \qquad z_{x*} = 0 \cdot 5(z_{\bar{x}} + z_x).$$

Throughout what follows we apply the conditions

$$\begin{array}{c} a \geqslant M_0 > 0, \quad \rho \geqslant M_0 > 0, \quad \mathcal{E}_s > 0, \quad \sigma_s \geqslant 0, \quad \sigma_1 + \sigma_2 \geqslant M_1 > 0, \\ | b_s | \leqslant M_2, \quad | \bar{b}_s | \leqslant M_2, \quad | d_s | \leqslant M_2, \quad s = 1, 2 \end{array} \tag{4}$$

which we include in the set-up of problem (1)–(4).

All the results obtained in this section also hold for the first boundary problem ($z_0 = z_N = 0$).

2. *A priori estimates when the coefficient a is "differentiable" with respect to* t($|a_{\bar{t}}| \leqslant M$). Problem (1)–(4) was considered in a somewhat less general statement in [4]. We shall therefore start with a description of the results of [4], which may be summarized as three theorems.

THEOREM 3. *If the conditions are fulfilled*: $0{\cdot}5 \leqslant \alpha_1 \leqslant 1, \varphi \equiv 0$,

$$|a_{\bar t}| \leqslant M_3,\ |(b_s)_x| \leqslant M_4,\ |\sigma_{s\bar t}| \leqslant M_5 \sqrt{\sigma_s \check\sigma_s} \qquad (s = 1, 2), \tag{5}$$

the solution of problem (1)–(4) is subject to the estimate, for sufficiently small $\tau \leqslant \tau_0$:

$$\|z\|_0 \leqslant M\Big[\sum_{t'=\tau}^{t'=t} \tau \overline{\|\psi(x, t')\|_2^2}\Big]^{\frac{1}{2}}, \tag{6}$$

where

$$\overline{\|\psi\|}_2 = \|\psi\|_2 + |\nu_1|/\sqrt{\mathcal{E}_1} + |\nu_2|/\sqrt{\mathcal{E}_2}. \tag{7}$$

In the case of the first boundary problem ($z_0 = 0, z_N = 0$), $\|\psi\|_2$ appears in (6) instead of $\overline{\|\psi\|}_2$.

The constant τ_0 depends only on $M_s (s = 0, 1, \ldots, 5)$.

THEOREM 4. *If conditions* (5) *are fulfilled*, $0{\cdot}5 \leqslant \alpha_1 \leqslant 1$, *and* $\psi \equiv 0$, *we have for* $\tau \leqslant \tau_0$:

$$\begin{aligned} &\|\sqrt{a(x,t)}\, z_{\bar x}\|_2 + \sqrt{\sigma_1(t)}\,|z_0| + \sqrt{\sigma_2(t)}\,|z_N| \leqslant \\ &\qquad \leqslant M(\|\sqrt{a(x,0)}\ \varphi_{\bar x}\| + \sqrt{\sigma_1(0)}\,|\varphi(0)| + \sqrt{\sigma_2(0)}\,\|\varphi(1)\|_2). \end{aligned} \tag{8}$$

If, in addition, $a \leqslant M_6, 0 \leqslant \sigma_s \leqslant M_1, s = 1, 2$, *we have*

$$\overline{\|z\|}_2 + \|z_{\bar x}\|_2 \leqslant M(\overline{\|\varphi\|}_2 + \|\varphi_{\bar x}\|_2), \tag{9}$$

where

$$\overline{\|z\|}_2 = \|z\|_2 + |z(0, t)| + |z(1, t)|. \tag{10}$$

Inequality (9) also holds for the first boundary problem, if $\overline{\|z\|}_2$ is replaced by $\|z\|_2$.

A proof of Theorems 3 and 4 is contained in [4]. We shall give the proof in a shortened and somewhat modified form, convenient for passage to a system of equations. We confine ourselves to the proof of Theorem 3.

We multiply equation (1) by $h\tau z_{\bar t}$ and sum over the net $\{x_i, 1 \leqslant i \leqslant N-1\}$. On making use of Green's formula (B. 22) and boundary conditions (2), we obtain

$$\tau[\rho, z_{\bar t}^2] + \alpha_1 I = \alpha_2 \check I + \tau(\Psi, z_{\bar t}) + R, \qquad \Psi = \psi + Q(z), \tag{11}$$

$$I = (a, z_{\bar x}^2] + \sigma_1 z_0^2 + \sigma_2 z_N^2, \qquad [\rho, z_{\bar t}^2] = (\rho, z_{\bar t}^2) + \mathcal{E}_1 z_{\bar t,0}^2 + \mathcal{E}_2 z_{\bar t,N}^2, \tag{12}$$

$$\begin{aligned} R = (\alpha_1 a - (1-\alpha_1)\check a,\ z_{\bar x}\check z_{\bar x}] + (\alpha_1\sigma_1 - \alpha_2\check\sigma_1) z_0 \check z + \\ + (\alpha_1\sigma_2 - \alpha_2\check\sigma_2) z_N \check z_N + \tau\nu_1 z_{\bar t,0} + \tau\nu_2 z_{\bar t,N}. \end{aligned} \tag{13}$$

In order to transform the expression

$$R_1 = (\alpha_1 a - \alpha_2 \check a,\ z_{\bar x}\check z_{\bar x}] = (2\alpha_1 - 1)(a, z_{\bar x}\check z_{\bar x}] + \alpha_2\tau(a_{\bar t}, z_{\bar x}\check z_{\bar x}]$$

we make use of the identities

$$a z_{\bar x}\check z_{\bar x} = a z_{\bar x}^2 - \underline{\tau a z_{\bar x} z_{\bar x\bar t}}, \tag{14}$$

$$a z_{\bar x}\check z_{\bar x} = \check a \check z_{\bar x}^2 + \tau a z_{\bar x\bar t} z_{\bar x} + \tau a_{\bar t}\check z_{\bar x}^2 = \check a \check z_{\bar x}^2 + \underline{\tau a z_{\bar x} z_{\bar x\bar t}} - \tau^2 a z_{\bar x\bar t}^2 + \tau a_{\bar t}\check z_{\bar x}^2. \tag{15}$$

It follows from this that

$$(2\alpha_1-1)az_{\bar{x}}\check{z}_{\bar{x}} = (\alpha_1-0\cdot 5)\{(az_{\bar{x}}^2+\check{a}\check{z}_{\bar{x}}^2)-\tau^2 az_{\bar{x}t}^2+\tau a_{\bar{t}}\check{z}_{\bar{x}}^2\}.$$

On similarly transforming the second and third terms in (13) and substituting the results obtained in (11), we obtain the fundamental identity

$$\tau[\rho, z_{\bar{t}}^2]+0\cdot 5I+(\alpha_1-0\cdot 5)\{(a, z_{\bar{x}t}^2]+\sigma_1 z_{\bar{t},0}^2+\sigma_2 z_{\bar{t},N}^2\}\tau^2 = \tag{16}$$
$$= 0\cdot 5\check{I}+(\alpha_1-0\cdot 5)\tau\{(a_{\bar{t}}, \check{z}_{\bar{x}}^2)+\sigma_{1\bar{t}}\check{z}_0^2+\sigma_{2\bar{t}}\check{z}_N^2\}+(1-\alpha_1)\tau\{(a_{\bar{t}}, z_{\bar{x}}\check{z}_{\bar{x}}]+\sigma_{1\bar{t}}z_0\check{z}_0+$$
$$+\sigma_{2\bar{t}}z_N\check{z}_N\}+\tau\{(\Psi, z_{\bar{t}})+\nu_1 z_{\bar{t},0}+\nu_2 z_{\bar{t},N}\}.$$

We estimate the expressions in the curly brackets on the right-hand side of (16) by using conditions (4), (5) and inequality (B. 15). We note here the inequality

$$|\sqrt{\sigma}| = |\sqrt{\check{\sigma}}+\tau\sigma_{\bar{t}}/(\sqrt{\sigma}+\sqrt{\check{\sigma}})| \leqslant \sqrt{\check{\sigma}}+M_5\sqrt{\sigma\check{\sigma}}/(\sqrt{\sigma}+\sqrt{\check{\sigma}}) \leqslant (1+M_5\tau)\sqrt{\check{\sigma}}.$$

As a result, we arrive at the integral inequality

$$\tau(\rho, z_{\bar{t}}^2)+I \leqslant (1+M\tau)\{\check{I}+\tau\overline{\|\Psi\|_2^2}\} \text{ for } \tau \leqslant \tau_0 = \bar{\tau}_0/\alpha_2. \tag{17}$$

We can readily obtain, by using (4), (5) and (1.16a):

$$\overline{\|\Psi\|_2} \leqslant \overline{\|\psi\|_2}+\overline{M}I+M\check{I}, \tag{18}$$

where

$$\overline{M} = M(\|\widetilde{b_{1x}}\|_0+\|\widetilde{b_1}\|_0+\|\widetilde{d_1}\|_0). \tag{19}$$

We can therefore write

$$\tau(\rho, z_{\bar{t}}^2)+I \leqslant (1+M\tau)(\check{I}+\tau\overline{\|\psi\|_2^2}) \qquad \text{for } \tau \leqslant \tau_0 = \bar{\tau}_0/\alpha_2(1+M\overline{M}). \tag{20}$$

It is clear from this that, when $b_1 = \bar{b}_1 = d \equiv 0$, the previous condition $\tau \leqslant \leqslant \bar{\tau}_0/\alpha_2$ holds, where $\bar{\tau}_0$ depends only on M_3, M_5, M_0 and M_1 and vanishes for $a_{\bar{t}} \equiv 0$, $\sigma_{s\bar{t}} \equiv 0$.

We obtain (6) by solving inequality (20) and taking (1.16a) into account. The proof of Theorem 4 makes use of (20), where we must write $\overline{\|\psi\|_2} = 0$ and take into account the initial condition $z(x, 0) = \varphi(x)$.

REMARKS.

1) The condition $\tau \leqslant \tau_0$ is lifted if $b_1 = \bar{b}_1 = d_1 \equiv 0$, $a_{\bar{t}} = \sigma_{1\bar{t}} = \sigma_{2\bar{t}} \equiv 0$ or $b_1 = \bar{b}_1 = d_1 \equiv 0$ and $\alpha_2 = 0$ $(\alpha_1 = 1)$.

2) Only Theorem 3 is used in the proof of convergence. It is not required in the proof of this that the coefficient $a(x, t)$ be bounded from above.

3) If the conditions of Theorem 4 are fulfilled, we have

$$\|z\|_0 \leqslant M\Big\{\Big[\sum_{t'=\tau}^{t'=t}\overline{\|\psi(x, t')\|_2^2}\Big]^{\frac{1}{2}}+\|\varphi_{\bar{x}}\|_2+|\varphi(0)|+|\varphi(1)|\Big\}. \tag{21}$$

4) Inequality (21) is obtained in [8] and [9] for the first boundary problem with $\alpha_1 = 1$ and the subsidiary assumption $|a_x| \leqslant M$, which makes it unsuitable for our purposes, since it excludes the possibility of fixed discontinuities of the coefficient of heat conduction.

5) Inequality (9) expresses the stability of the solution of problem (1)–(4) with respect to the initial data in the sense of the norm $\|\bar{z}\|_2+\|z_{\bar{x}}\|_2$ (cf. [8]).

6) Theorems 3, 4, as also Theorems 5 and 6 below, retain their force in the case of the more general boundary condition

$$(a^{(+1)}z_x)^{(\alpha)} = \mathcal{E}_1 z_{\bar{t}} + (\sigma_1 z + h\zeta_1 z + h\eta_1 z_x)^{(\alpha)} - \nu_1 \quad \text{for } x = 0,$$

where $|\zeta_1| \leqslant M$, $|\eta_1| \leqslant M$, and of the analogous condition with $x = 1$.

3. *A priori estimates when the coefficients a, b_1 and b_2 are bounded.* Special difficulties appear when estimating the solution of problem (1)–(4) in the case when conditions (5) are not fulfilled and the coefficients a, b_1 and b_2 are merely bounded. We have succeeded in obtaining an effective *a priori* estimate only for the case $\alpha_1 = 1$. We shall state the corresponding theorem, proved in [4], for the case $\varphi \equiv 0$.

THEOREM 5. *If $\alpha_1 = 1$, and*

$$|\rho_{\bar{t}}| \leqslant M, \qquad |\mathcal{E}_{s\bar{t}}| \leqslant M\mathcal{E}_s, \qquad s = 1, 2, \tag{22}$$

the solution of problem (1)–(4) *is subject to the inequalities, for sufficiently small* $\tau \leqslant \tau_0$:

$$[1, z^p]^{\frac{1}{p}} \leqslant M(p)\Big[\sum_{t'=\tau}^{t'=t} \tau \|\varphi(x, t')\|_{\xi}^{\xi}\Big]^{\frac{1}{p}}, \tag{23}$$

$$\Big(\sum_{t'=\tau}^{t'=t} \tau \|z(x, t')\|^p\Big)^{\frac{1}{p}} \leqslant M(p)\Big[\sum_{t'=\tau}^{t'=t} \tau \|\psi(x, t')\|_{\xi}^{\xi}\Big]^{\frac{1}{p}}, \tag{24}$$

where $p = 2^n, n \geqslant 1$ *is any integer,* $M(p) = M(n) = M\sqrt{n}2^n e^{Mn2^n}$, $\tau_0 = M(1+ \|\tilde{b}_1\|_0/n \cdot 2^n)$ *is independent of n only when* $b_1 \equiv 0$.

The proof of Theorem 5 makes use of a difference equation for the function z^{2n} and the integral inequality obtained for this equation in [4].

4. *An improved a priori estimate for a six-point equation.* We consider the following problem:

$$\rho z_{\bar{t}} - (az_{\bar{x}})_x^{(\alpha)} = \Psi^{(\alpha)}, \tag{25}$$

$$\left.\begin{aligned} &(a^{(+1)}z_x)^{(\alpha)} = \mathcal{E}_1 z_{\bar{t}} + (\sigma_1 z)^{(\alpha)} - \nu_1^{(\alpha)} && \text{for } x = 0,\\ &-(az_{\bar{x}})^{(\alpha)} = \mathcal{E}_2 z_{\bar{t}} + (\sigma_2 z)^{(\alpha)} - \nu_2^{(\alpha)} && \text{for } x = 1,\\ &z(x, 0) = 0, \end{aligned}\right\} \tag{26}$$

where

$$\Psi = Q(z) + \psi, \qquad Q(z) = bz_{\bar{x}} + b^{(+1)}z_x + dz. \tag{27}$$

The coefficients of the problem satisfy the conditions:

$$\left.\begin{aligned} &0 < M_1 \leqslant a \leqslant M_2,\ M \geqslant \rho \geqslant M_1 > 0,\ |b| \leqslant M,\ |d| \leqslant M,\quad \sigma_s \geqslant 0,\\ &\sigma_1+\sigma_2 \geqslant M > 0,\ |a_{\bar{t}}| \leqslant M,\ |\sigma_{s\bar{t}}| \leqslant M\sqrt{\sigma_s\check{\sigma}_s},\ 0 < \mathcal{E}_s \leqslant M \quad (s = 1, 2), \end{aligned}\right\} \tag{28}$$

where one of the cases holds:

$$0 \leqslant \sigma_s \leqslant M \text{ or } \sigma_s = \infty \text{ (boundary condition of the first kind).} \tag{29}$$

THEOREM 6. *If $z(x, t)$ is a solution of problem (25)–(29) and $0{\cdot}5 \leqslant \alpha_1 \leqslant 1$, we have for sufficiently small $\tau \leqslant \tau_0$:*

$$\|z\|_0 \leqslant M\{\|\psi(x, t)\|_{5*} + \|\psi(x, 0)\|_5 +$$

$$+[\sum_{t'=\tau}^{t'=t} \tau(\|\psi(x, t')\|_{5*}^2 + \|\psi_{\bar{t}}(x, t')\|_{5*}^2)]^{\frac{1}{2}}\} + \bar{M}[\sum_{t'=\tau}^{t'=t} \tau\|\psi(x, t')\|_5^2]^{\frac{1}{2}}, \quad (30)$$

where $\bar{M} = 0$ for $b \equiv 0$.

We write z as the sum $z = v+w$, where w is the solution of problem (1.1)–(1.4) for $d \equiv 0$, and v is defined from the conditions:

$$\left.\begin{aligned} &\rho v_{\bar{t}} - (a v_{\bar{x}})_x^{(\alpha)} = Q^{(\alpha)}(v) - [\rho w_{\bar{t}} - Q^{(\alpha)}(w)], && \\ &(a^{(+1)} v_x)^{(\alpha)} = \mathcal{E}_1 v_{\bar{t}} + (\sigma_1 v)^{(\alpha)} + \mathcal{E}_1 w_{\bar{t}} && \text{for } x = 0, \\ &-(a v_{\bar{x}})^{(\alpha)} = \mathcal{E}_2 v_{\bar{t}} + (\sigma_2 v)^{(\alpha)} + \mathcal{E}_2 w_{\bar{t}} && \text{for } x = 1, \\ &v(x, 0) = -w(x, 0). && \end{aligned}\right\} \quad (31)$$

By Theorem 1 and Lemma 5:

$$\|w\|_0 \leqslant M\|\psi\|_{5*}, \quad \|w_{\bar{x}}\|_1 \leqslant M\|\psi\|_{5*}, \quad \|w_{\bar{t}}\|_0 \leqslant M(\|\psi_{\bar{t}}\|_{5*} + \|\psi\|_{5*}). \quad (32)$$

On now using Lemma 4, we find that

$$\|Q(w)\|_2 \leqslant \bar{M}\|w_{\bar{x}}\|_2 + M\|w\|_2 \leqslant \bar{M}\|\psi\|_5 + M\|\psi\|_{5*}. \quad (33)$$

We make use of inequality (21) for estimating the solution of problem (31). The norm

$$\|\rho w_{\bar{t}} - Q^{(\alpha)}(w)\|_2 + \sqrt{\mathcal{E}_1}\,|w_{\bar{t},0}| + \sqrt{\mathcal{E}_2}\,|w_{\bar{t},N}| \leqslant \bar{M}\|\psi\|_5 + M\{\|\psi\|_{5*} + \|\psi_{\bar{t}}\|_{5*}\},$$

appears on the right-hand side of (21), and we estimate this with the aid of inequalities (32) and (33). This leads us to (30), in view of (21) and the inequality $\|z\|_0 \leqslant \|v\|_0 + \|w\|_0$.

Direct use is made of Theorem 6 in proving a theorem on the convergence of homogeneous difference schemes for an equation of the parabolic type in the case of fixed discontinuities of the heat conduction coefficient.

Distinguishing the solution of the "stationary" problem (1.1)–(1.4) by the root method improves the *a priori* estimate for our problem, by enabling us to introduce the norms $\|\psi\|_5$ and $\|\psi\|_{5*}$.

It must be mentioned that the most accurate estimate is obtained when $b(x, t) \equiv 0$ and $\psi(x, 0) = 0$, since only $\|\psi\|_{5*}$ appears on the right-hand side of inequality (30):

$$\|z\|_0 \leqslant M\{\|\psi(x, t)\|_{5*} + [\sum_{t'=\tau}^{t'=t} \tau\|\psi(x, t')\|_{5*}^2 + \|\psi_{\bar{t}}(x, t')\|_{5*}^2]^{\frac{1}{2}}\}. \quad (34)$$

We illustrated the merit of the norm $\|\psi\|_{5*}$ with the aid of an example in § 1, Section 2.

To prove theorems on convergence and accuracy, we write the solution of problem (1)–(4) as the sum $\bar{z}+\bar{\bar{z}}$, where $\bar{z}$ is the solution of the same problem with homogeneous boundary conditions and a good right-hand side $\bar{\psi}$ (i.e. having a high order of smallness with respect to h and τ), whilst $\bar{\bar{z}}$ is the solution of problem (25)–(29) with a poor right-hand side $\bar{\bar{\psi}}^{(\alpha)}$. The function $\bar{z}$ is estimated with the aid of inequality (6), whilst Theorem 6 is used for estimating $\bar{\bar{z}}$.

We must also mention an *a priori* estimate (in the mean), obtained in [12] for a six-point scheme with the sole assumption that $a(x, t)$ is bounded. If $z(x, t)$ is a solution of equation (25) with the conditions

$$z_0 = z_N = 0, \quad z(x,\ 0) = \varphi(x), \quad 0 < M_0 \leqslant a \leqslant M_1, \quad \rho \geqslant M_0 > 0,$$
$$|\rho_{\bar{t}}| \leqslant M, \ |b| \leqslant M, \ |d| \leqslant M,$$

then

$$\| z \|_2 \leqslant M \Big[\sum_{t'=\tau}^{t'=t} \tau \| \psi(x,\ t') \|_2^2 \Big]^{\frac{1}{2}} + M(1+\gamma) \| \varphi \|_2 \qquad (\gamma = \tau / h^2). \tag{35}$$

5. *Systems of equations.* We now take a system of equations which is the difference analogue of a system of linear equations of the parabolic type. Let $\mathbf{z} = \{z^j, j = 1, 2, \dots, r\}$ be a vector function given on the net $\bar{\Omega}$ and satisfying the conditions

$$\left.\begin{array}{ll} \boldsymbol{\rho} \mathbf{z}_{\bar{t}} - (\mathbf{a} \mathbf{z}_{\bar{x}})_x^{(\alpha)} = \boldsymbol{\Psi}^{(\alpha)} & \text{on } \Omega_1, \\ \mathbf{z} = 0 \quad \text{for } x = 0,\ x = 1 \text{ and } t = 0, & \end{array}\right\}, \tag{36}$$

where $\mathbf{a} = (a_{ij})$ is a symmetric positive definite matrix; $\rho = (\rho_{ij})$ is a positive definite matrix, i.e. to be precise,

$$a_{ij} = a_{ji}, \sum_{i,j=1}^{r} a_{ij} \xi_i \xi_j \geqslant \beta_1 \sum_{j=1}^{r} \xi_j^2, \sum_{i,j=1}^{r} \rho_{ij} \xi_i \xi_j \geqslant \beta_2 \sum_{j=1}^{r} \xi_j^2, \ \beta_1 \geqslant M > 0, \ \beta_2 \geqslant M > 0;$$

ξ_j are any real numbers; β_1 and β_2 are constants independent of h and τ;

$$\boldsymbol{\Psi} = \mathbf{Q}(\mathbf{z}) + \boldsymbol{\psi}, \qquad \mathbf{Q}(\mathbf{z}) = \mathbf{b} \mathbf{z}_{\bar{x}} + \mathbf{b}^{(+1)} \mathbf{z}_x + \mathbf{d} \mathbf{z}, \tag{37}$$

$\mathbf{b} = (b_{ij})$, $\mathbf{d} = (d_{ij})$ is a matrix, $\boldsymbol{\psi} = (\psi^j)$, $\mathbf{z} = (z^j)$ are vectors.

In addition, the following conditions are satisfied:

$$|a_{ij}| \leqslant M, \qquad |b_{ij}| \leqslant M, \qquad |d_{ij}| \leqslant M, \qquad |(a_{ij})_{\bar{t}}| \leqslant M. \tag{38}$$

To simplify the treatment, we shall confine ourselves to the case of the first boundary problem, although the results below can be extended to the case of boundary conditions of a more general type, analogous to conditions (2) for a single equation (1). We have.

THEOREM 7. *If $\alpha_1 \geqslant 0{\cdot}5$ and $\tau \leqslant \tau_0$ is sufficiently small, the solution of problem* (36)–(38) *satisfies the inequality*

$$\| \mathbf{z} \|_0 \leqslant M \Big[\sum_{t'=\tau}^{t'=t} \tau \| \psi(x,\ t') \|_2^2 \Big]^{\frac{1}{2}}, \tag{39}$$

where

$$\|\mathbf{z}\|_0 = \max_{(x)} \sqrt{\sum_{j=1}^{r}(z^j)^2}, \quad \|\psi\|_2 = \Big(\sum_{i=1}^{r} (\psi^j)^2,\ 1\Big)^{\frac{1}{2}}, \tag{40}$$

The theorem is proved by analogy with Theorem 3. We form the scalar product of equation (36) with the vector $h\tau\mathbf{z}_{\bar{t}}$ and sum over the net $\{x = h, 2h, \ldots, (N-1)h = 1-h\}$:

$$\tau(\boldsymbol{\rho}\mathbf{z}_{\bar{t}},\ \mathbf{z}_{\bar{t}})+\alpha_1 I = \alpha_2\check{I}+\tau(\boldsymbol{\Psi}^{(\alpha)},\ \mathbf{z}_{\bar{t}})+((\alpha_1\mathbf{a}-\alpha_2\check{\mathbf{a}})\mathbf{z}_{\bar{x}},\ \check{\mathbf{z}}_{\bar{x}}], \tag{41}$$

where

$$I = (\mathbf{a}\mathbf{z}_{\bar{x}},\ \mathbf{z}_{\bar{x}}] = \sum_{ij} (a_{ij}z^i_{\bar{x}},\ z^j_{\bar{x}}],$$

We transform the expression

$$((\alpha_1\mathbf{a}-\alpha_2\check{\mathbf{a}})\mathbf{z}_{\bar{x}},\ \check{\mathbf{z}}_{\bar{x}}] = \sum_{i,\ j} ((\alpha_1 a_{ij}-\alpha_2\check{a}_{ij})z^i_{\bar{x}},\ \check{z}^j_{\bar{x}}]$$

$$= (2\alpha_1-1)\sum_{i,\ j} (a_{ij}z^j_{\bar{x}}\ .\ \check{z}^i_{\bar{x}}]+\alpha_2\tau\sum_{i,\ j} ((a_{ij})_{\bar{t}}\, z^j_{\bar{x}},\ \check{z}^i_{\bar{x}}].$$

We can write, by analogy with Section 2:

$$a_{ij}z^i_{\bar{x}}\check{z}^j_{\bar{x}} = a_{ij}\check{z}^i_{\bar{x}}z^j_{\bar{x}}-\underline{\tau a_{ij}z^i_{\bar{x}}z^j_{\bar{x}\bar{t}}} \qquad \text{for } j < i,$$

$$a_{ij}z^i_{\bar{x}}\check{z}^j_{\bar{x}} = \check{a}_{ij}\check{z}^i_{\bar{x}}z^j_{\bar{x}}+\underline{\tau a_{ij}z^i_{\bar{x}\bar{t}}z^j_{\bar{x}}}-\tau^2 a_{ij}z^i_{\bar{x}\bar{t}}z^j_{\bar{x}t}+\tau(a_{ij})_{\bar{t}}(\check{z}^i_{\bar{x}}\check{z}^j_{\bar{x}} \qquad \text{for } j > i,$$

$$a_{ii}z^i_{\bar{x}}\check{z}^i_{\bar{x}} = \tfrac{1}{2}[a_{ii}(z^i_{\bar{x}})^2+\check{a}_{ii}(\check{z}^i_{\bar{x}})^2]-\tfrac{1}{2}\tau^2 a_{ii}(z^t_{\bar{x}\bar{t}})^2+\frac{\tau}{2}(a_{ii})_{\bar{t}}(\check{z}^i_{\bar{x}})^2 \qquad \text{for } j = i.$$

On using the symmetry of the matrix (a_{ij}), we get

$$(\mathbf{a}\mathbf{z}_{\bar{x}},\check{\mathbf{z}}_{\bar{x}}] = \sum_{i,j}(a_{ij}z^i_{\bar{x}},\ \check{z}^j_{\bar{x}}] = \tfrac{1}{2}(I+\check{I})-\frac{\tau^2}{2}(\mathbf{a}\mathbf{z}_{\bar{x}\bar{t}},\mathbf{z}_{\bar{x}\bar{t}}]+\frac{\tau}{2}(\mathbf{a}_{\bar{t}}\check{\mathbf{z}}_{\bar{x}},\ \check{\mathbf{z}}_{\bar{x}}]. \tag{42}$$

Returning to (41), we derive the integral identity

$$\tau(\boldsymbol{\rho}\mathbf{z}_{\bar{t}},\ \mathbf{z}_{\bar{t}}]+0{\cdot}5I+(\alpha_1-0{\cdot}5)(\mathbf{a}\mathbf{z}_{\bar{x}\bar{t}},\ \mathbf{z}_{\bar{x}\bar{t}}] \tag{43}$$

$$= 0{\cdot}5\check{I}+(\alpha_1-0{\cdot}5)\tau(\mathbf{a}_{\bar{t}}\check{\mathbf{z}}_{\bar{x}},\check{\mathbf{z}}_{\bar{x}}]+\alpha_2\tau(\mathbf{a}_{\bar{t}}\mathbf{z}_{\bar{x}},\ \check{\mathbf{z}}_{\bar{x}}]\,\tau(\Psi,\mathbf{z}_{\bar{t}}).$$

Proceeding now as in Section 2, and taking into account

$$\|\mathbf{z}\|_0 \leqslant M\|\mathbf{z}_{\bar{x}}\|_2, \tag{44}$$

we arrive at inequality (39).

Theorem 8. *If* $\alpha_1 \geqslant 0{\cdot}5$ *and* τ is sufficiently small ($\tau \leqslant \tau_0$), then

$$\|\mathbf{z}\|_0 \leqslant M\{\|\boldsymbol{\psi}(x,\ t)\|_3+\|\psi(x,0)\|_3+[\sum_{t'=\tau}^{t'=t}\tau\|\boldsymbol{\psi}(x\ t')\|_3^2+\tau\|\boldsymbol{\psi}_{\bar{t}}(x,\ t')\|_3^2]^{\frac{1}{2}}\}+$$

$$+\overline{M}[\sum_{t'=\tau}^{t'=t}\tau\|\psi(x,t')\|_3^2]^{\frac{1}{2}}, \tag{45}$$

where $\overline{M} = 0$ *for* $b \equiv 0$.

This theorem is similar to Theorem 5. The proof of it uses Theorem 2 and Lemma 7.

6. *Differential-difference equations.* Analogous *a priori* estimates hold for the solutions of a differential equation of the parabolic type and differential-difference equations obtained from it by Rothe's method [10] (replacement of the derivative with respect to t by a difference quotient) and by the straight-line method (replacement of the operator with respect to x by a difference expression).

A. *Rothe's method.* Let $z(x, t)$ (netted with respect to t), defined in the domain $(0 \leqslant x \leqslant 1)\times\omega_\tau^T = \{t_j = j_\tau, 0 \leqslant j \leqslant L\}$, be a solution of the differential-difference equation

$$\rho(x,t)z_{\bar{t}} - \frac{d}{dx}\left(a(x,t)\frac{dz}{dx}\right) = Q(z)+\psi \qquad (0 < x < 1, \quad t = \tau, 2\tau, \ldots, L\tau) \tag{46}$$

with the conditions

$$\begin{aligned} z(x,0) = 0, \qquad k\frac{dz}{dx} &= \mathcal{E}_1 z_{\bar{t}} + \sigma_1 z - \nu_1(t) \qquad \text{for } x = 0, \\ -k\frac{dz}{dx} &= \mathcal{E}_2 z_{\bar{t}} + \sigma_2 z - \nu_2(t) \qquad \text{for } x = 1, \end{aligned} \tag{47}$$

$$Q(z) = \frac{d}{dx}(bz) + dz, \tag{48}$$

$$\begin{gathered} a \geqslant M > 0, \qquad \rho \geqslant M > 0, \qquad |b| \leqslant M, \qquad |d| \leqslant M, \qquad \mathcal{E}_s > 0, \\ \sigma_s \geqslant 0, \qquad \sigma_1+\sigma_2 \geqslant M > 0, \qquad s = 1, 2. \end{gathered} \tag{49}$$

If, in addition, conditions (5) are fulfilled, where we have to write $|db/dx| \leqslant M$ instead of $|b_{sx}| \leqslant M$, the following inequalities hold for the solution of problem (46)–(49) when $\tau \leqslant \tau_0$:

$$\|z\|_0 \leqslant M\Big(\sum_{t'=\tau}^{t'=t} \tau \overline{\|\psi(x,t')\|_2^2}\Big)^{\frac{1}{2}}, \tag{50}$$

$$\begin{aligned} \|z\|_0 \leqslant M\Big\{&\|\psi(x,t)\|_{5^*} + \|\psi(x,0)\|_5 + \\ &+\Big[\sum_{t'=\tau}^{t'=t} \tau(\|\psi(x,t')\|_{5^*} + \|\psi_{\bar{t}}(x,t')\|_{5^*})^2\Big]^{\frac{1}{2}}\Big\} + \bar{M}\Big[\sum_{t'=\tau}^{t'=t} \tau\|\psi(x,t')\|_5^2\Big]^{\frac{1}{2}}, \end{aligned} \tag{51}$$

where

$$\|z\|_0 = \max_{0<x\leqslant 1}|z|, \qquad \|\psi\|_m = \Big[\int_0^1 |\psi|^m dx\Big]^{\frac{1}{m}}, \qquad m = 1, 2, \quad \bar{M} = 0 \quad \text{for } b \equiv 0$$

$$\|\psi\|_5 = \Big\|\int_0^x \psi d\xi\Big\|_2 + \Big|\int_0^1 \psi dx\Big| + |\nu_1| + |\nu_2|, \quad \overline{\|\psi\|_2} = \|\psi\|_2 + |\nu_1|/\sqrt{\mathcal{E}_1} + |\nu_2|/\sqrt{\mathcal{E}_2}$$

$$\|\psi\|_{5^*} = \Big\|\int_0^x \psi d\xi\Big\|_1 + \Big|\int_0^1 \psi dx\Big| + |\nu_1| + |\nu_2|.$$

If we have only conditions (49) and $|\rho_{\bar{t}}| \leqslant M$, $|\mathcal{E}_{s\bar{t}}| \leqslant M\mathcal{E}_s$, an analogue of Theorem 4 holds.

B. *Method of straight lines.* Let $z(x, t)$ denote a function defined in the domain $(0 \leqslant t \leqslant T) \times \omega'_h\{x = 0, h, 2h, \ldots, Nh = 1\}$ (i.e. netted with respect to x) and satisfying the equation

$$\rho(x, t)\frac{\mathrm{d}z}{\mathrm{d}_t} - (az_{\bar{x}})_x = \psi + Q(z), \qquad Q(z) = (b_1 z^*)_x + \bar{b}_1 z_{x*} + dz, \tag{52}$$

and the conditions

$$\left.\begin{aligned} z(x,\ 0) = 0,\ a_1^{(+1)} z_x &= \mathcal{E}_1 \frac{\mathrm{d}z}{\mathrm{d}t} + \sigma_1 z - \nu_1 \qquad \text{for } x = 0, \\ -az_x &= \mathcal{E}_2 \frac{\mathrm{d}z}{\mathrm{d}t} + \sigma_2 z - \nu_2 \qquad \text{for } x = 1. \end{aligned}\right. \tag{53}$$

If conditions (4) are fulfilled, together with $|\mathrm{d}a/\mathrm{d}t| \leqslant M$, $|\mathrm{d}\sigma_s/\mathrm{d}t| \leqslant M\sigma\sqrt{\delta_s}$, $|b_{1x}| \leqslant M$, we have

$$\| z \|_0 \leqslant \overline{\| z \|}_2 + \| z_{\bar{x}} \|_2 \leqslant M \left\{ \left[\int_0^t \overline{\| \psi(x, t) \|}_2^2 \mathrm{d}t' \right]^{\frac{1}{2}} + \overline{\| \varphi \|}_2 + \| \varphi_{\bar{x}} \|_2 \right\}, \tag{54}$$

where all the norms have the same meaning as in Section 2.

There is no difficulty in writing down the other *a priori* estimates, obtained above for problem (1)–(4), and also estimates for the solutions of the system of differential-difference and differential equations.

All these estimates enable us to prove the convergence of Rothe's method and of the straight line method in the class of discontinuous coefficients, both for the case of fixed and for the case of moving discontinuities.

REMARK. All the results of this section can be carried over to the case of non-uniform nets if, as indicated in § 1, account is taken of the alteration in a non-uniform net of the meaning of the norms appearing in the *a priori* estimates. For instance, in inequality (30) we have to replace $\|\psi\|_5$ and $\|\psi\|_5^*$ by the norms $\|\psi\|_5^*$ and $\|\psi\|^*_{5*}$, which are defined in Section 3. Introduction, and we have to remember that τ is a net function in the sum over the time net.

§ 3. EQUATIONS OF THE HYPERBOLIC TYPE

1. *The difference problem.* A differential equation of the hyperbolic type

$$\mathcal{L}_2 u = c(x, t)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(k(x, t)\frac{\partial u}{\partial x}\right) - Q(u) = \psi \qquad 0 < x < 1, 0 < t \leqslant T), \tag{1}$$

$$Q(u) = b\frac{\partial u}{\partial x} + c_1 \frac{\partial u}{\partial t} + qu \tag{2}$$

has the following equation as its difference analogue:

$$\left.\begin{aligned} \mathcal{P}_2 z &= \rho z_{\bar{t}\bar{t}} - (az_{\bar{x}})_x^{(\alpha)} - Q(z) = \psi \qquad \text{on } \Omega_2, \\ Q(z) &= (bz_{x*} + dz)^{(\alpha)} + gz_{\bar{t}} + \check{g}\check{z}_{\bar{t}}, \end{aligned}\right\} \tag{3}$$

where the index α indicates summation over the three rows $t, t-\tau, t-2\tau$ of the net Ω_2 with weighting factors $\alpha_1, \alpha_2, \alpha_3$, such that

$$v^{(\alpha)} = \alpha_1 v + \alpha_2 \check{v} + \alpha_2 \check{\check{v}}, \qquad \alpha_1 \geqslant 0, \qquad \alpha_2 \geqslant 0, \qquad \alpha_3 \geqslant 0, \qquad \alpha_1+\alpha_2+\alpha_3 = 1.$$

Let the function $z(x, t)$, defined on $\overline{\Omega}$, be a solution of the problem

$$\mathcal{P}_2 z = \psi \text{ on } \Omega_2, \tag{4}$$

$$\left.\begin{aligned} l_1 z &= (a^{(+1)} z_x)^{(\alpha)} - \mathcal{E}_1(t)(z_{\bar{t}\bar{t}} + c_1 z_{\bar{t}} + \check{c}_1 \check{z}_{\bar{t}}) - (\sigma_1 z)^{(\alpha)} = -\nu_1 \\ &\qquad \text{for } x = 0,\ t = 2\tau, \ldots, T, \\ l_2 z &= (a z_{\bar{x}})^{(\alpha)} + \mathcal{E}_2(t)(z_{\bar{t}\bar{t}} + c_2 z_{\bar{t}} + \check{c}_2 \check{z}_{\bar{t}}) + (\sigma_2 z)^{(\alpha)} = \nu_2 \quad \text{for } x = 1, \end{aligned}\right\} \tag{5}$$

$$z(x, 0 = \varphi(x), \qquad z_t(x, 0) = \overline{\varphi}(x), \tag{6}$$

$$\left.\begin{aligned} &0 < M_0 \leqslant a \leqslant M_1, \quad 0 < M_0 \leqslant \rho \quad |a_{\bar{t}}| \leqslant M, \quad |\rho_t| \leqslant M, \\ &\qquad |b| \leqslant M, \quad |d| \leqslant M, \quad |g| \leqslant M, \\ &\mathcal{E}_s > 0, \quad \sigma_s \geqslant 0, \quad \sigma_1+\sigma_2 \geqslant M > 0, \quad |\sigma_{s\bar{t}}| \leqslant M\sqrt{\sigma_s \check{\sigma}_s}, \\ &\qquad |\mathcal{E}_{s\bar{t}}| \leqslant M, \quad |c_s| \leqslant M, \quad s = 1, 2. \end{aligned}\right\} \tag{7}$$

2. *An integral inequality.* To simplify the discussion we present an analysis for boundary conditions of the 1st kind $z_0 = 0$, $z_N = 0$. The transformation for boundary conditions (5) is carried out by analogy with [4] and Section 2 § 2.

We multiply equation (3) by $h\tau(z_{\bar{t}} + \beta \check{z}_{\bar{t}})$, where β is a parameter, and we sum over the net $\{x = h, 2h, \ldots, (N-1)h = 1-h\}$. On using Green's first formula (B. 22) and taking the boundary conditions into account ($z_0 = z_N = 0$), and also

$$z_{\bar{t}} z_{\bar{t}\bar{t}} = \frac{1}{2}(z_{\bar{t}}^2)_{\bar{t}} + \frac{\tau}{2} z_{\bar{t}\bar{t}}^2, \qquad \check{z}_{\bar{t}} z_{\bar{t}\bar{t}} = \frac{1}{2}(z_{\bar{t}}^2)_{\bar{t}} - \frac{\tau}{2} z_{\bar{t}\bar{t}}^2,$$

we get the identity

$$\begin{aligned} &\tau \frac{1+\beta}{2}(\rho, z_{\bar{t}}^2)_{\bar{t}} + \frac{1-\beta}{2}\tau(\rho, z_{\bar{t}\bar{t}}^2) + \alpha_1 I - \alpha_2(1-\beta)\check{I} - \beta\alpha_3 \check{\check{I}} \\ &= \tau(\Psi, z_{\bar{t}} + \beta \check{z}_{\bar{t}}) + (\alpha_1(1-\beta)a - \alpha_2 \check{a}, z_{\bar{x}} \check{z}_{\bar{x}}] + (\alpha_1 \beta a - \alpha_3 \check{\check{a}}, z_{\bar{x}} \check{\check{z}}_{\bar{x}}] + \\ &\quad + (\alpha_2 \beta \check{a} + \alpha_3(1-\beta)\check{\check{a}}, \check{z}_{\bar{x}} \check{\check{z}}_{\bar{x}}] - \frac{1+\beta}{2}\tau(\rho_{\bar{t}}, \check{z}_{\bar{t}}^2), \end{aligned} \tag{8}$$

where

$$I = (a, z_{\bar{x}}^2], \qquad \Psi = Q(z) + \psi. \tag{9}$$

Suppose now that $\beta = 1$, $\alpha_2 = 0$, $\alpha_1 \geqslant 0{\cdot}5$. Identity (8) can now be written in the form

$$\begin{aligned} &\tau(\rho, z_{\bar{t}}^2)_{\bar{t}} + 0{\cdot}5 I + 4\tau^2(\alpha_1 - 0{\cdot}5)(a, z_{\bar{x}t*}^2] = 0{\cdot}5 \check{\check{I}} + \\ &+ (2\alpha_1 - 1)\tau(a_{\bar{t}}, \check{z}_{\bar{x}}^2] + 2\alpha_3 \tau(a_{t*}, z_{\bar{x}} \check{\check{z}}_{\bar{x}}] + \tau(\Psi, z_{\bar{t}} + \check{z}_{\bar{t}}) - \tau(\rho_{\bar{t}}, z_{\bar{t}}^2), \end{aligned} \tag{10}$$

where

$$a_{t*} = \frac{a - \check{\check{a}}}{2\tau} = 0{\cdot}5(a_{\bar{t}} + \check{a}_{\bar{t}}).$$

On observing that

$$|(a_{t*}, z_{\bar{x}}\check{\check{z}}_{\bar{x}}]| \leqslant M(I+\check{\check{I}}), \qquad |\tau(\Psi, z_{\bar{t}}+\check{z}_{\bar{t}})| \leqslant M\tau\|\Psi\|_2^2+\frac{\tau}{T}(\rho, z_{\bar{t}}^2)+\frac{\tau}{T}(\check{\rho}, \check{z}_{\bar{t}}^2),$$

$$\tau|(\rho_{\bar{t}}, z_{\bar{t}}^2)| \leqslant M\tau(\rho, z_{\bar{t}}^2),$$

we have for sufficiently small $\tau \leqslant \tau_0$:

$$E \leqslant (1+M\tau)(\check{E}+M\tau\|\Psi\|_2^2), \tag{11}$$

where

$$E = (\rho, z_{\bar{t}}^2)+0\cdot5(I+\check{I}). \tag{12}$$

In view of conditions (7), it may easily be shown that

$$\|\Psi\|_2^2 \leqslant M\{\|\psi\|_2^2+(\|z_{\bar{x}}\|_2^2+\|z\|_2^2)^{(\alpha)}+\|z_{\bar{t}}\|_2^2+\|\check{z}_{\bar{t}}\|_2^2\}$$
$$\leqslant M[I+\check{I}+(\rho, z_{\bar{t}}^2)+(\check{\rho}, \check{z}_{\bar{t}}^2)].$$

As a result, we arrive at the following integral (energy) inequality (cf. [13]):

$$E \leqslant (1+M\tau)(\check{E}+\tau\|\psi\|_2^2). \tag{13}$$

For the general boundary conditions (5), we obtain the analogous inequality

$$E \leqslant (1+M\tau)(\check{E}+\tau\|\psi\|_2^2), \tag{14}$$

in which

$$\left.\begin{aligned} &E = [\rho, z_{\bar{t}}^2]+0\cdot5(I+\check{I}), \quad [\rho, z_{\bar{t}}^2] = (\rho, z_{\bar{t}}^2)+\mathcal{E}_1 z_{\bar{t},0}^2+\mathcal{E}_2 z_{\bar{t},N}^2,\\ &I = (a, z_{\bar{x}}^2]+\sigma_1 z_0^2+\sigma_2 z_N^2, \quad \overline{\|\psi\|_2} = \|\psi\|_2+|\nu_1|/\sqrt{\mathcal{E}_1}+|\nu_2|/\sqrt{\mathcal{E}_2}.\end{aligned}\right\} \tag{15}$$

Inserting $\beta = 0$, $\alpha_3 = 0$ in identity (5), we obtain the integral inequality (13) or (14), in which

$$E = [\rho, z_{\bar{t}}^2]+0\cdot5I. \tag{16}$$

3. *A priori estimates.* We make use of inequality (14) to estimate $\|z\|_0$. Inequality (1.16a) is required here. On solving (14), we get

$$E(t) \leqslant M\Big(E(\tau)+\sum_{t'=2\tau}^{t'=t}\tau\overline{\|\psi(x, t')\|_2^2}\Big). \tag{17}$$

We have from this, along with (6):

THEOREM 8. *If* $\alpha_2 = 0$, $\alpha_1 \geqslant 0\cdot5$ (*or* $\alpha_3 = 0$, $\alpha_1 \geqslant 0\cdot5$), given sufficiently small $\tau \leqq \tau_0$, we have the following estimate for the solution of problem (4)–(7):

$$\|z\|_0 \leqslant \tilde{M}\|z\|_7 \leqslant M\Big\{\|z^1\|_7+\Big[\sum_{t'=2\tau}^{t'=t}\tau\overline{\|\psi(x, t')\|_2^2}\Big]^{\frac{1}{2}}\Big\}, \tag{18}$$

where $\tilde{M} = \frac{1}{2}$, if $z_0 = 0$ or $z_N = 0$,

$$\|z\|_7 = \|z\|_2+\|\check{z}\|_2+\|z_{\bar{t}}\|_2+\|z_{\bar{x}}\|_2+\|\check{z}_{\bar{x}}\|_2, \quad z^0 = \varphi, \quad z^1 = \bar{\varphi}+\tau\varphi. \tag{19}$$

In the case $\alpha_3 = 0$, $\alpha_1 \geqslant 0\cdot5$, we have

$$\|z\|_7 = \|z\|_2+\|z_{\bar{x}}\|_2+\|z_{\bar{t}}\|_2.$$

It may easily be observed that (18) implies the inequality

$$\| z \|_0 \leqslant M\{(1+\gamma)(\| \varphi \|_2+\| \bar{\varphi} \|_2)+[\sum_{t'=2\tau}^{t'=t} \tau \overline{\| \psi(x, t') \|_2^2}]^{\frac{1}{2}}\}, \qquad \gamma = \frac{\tau}{h}. \quad (20)$$

We now take the problem

$$\mathcal{P}_2 z = \psi^{(\alpha)} \text{ on } \Omega_2, \quad (21)$$

$$l_1 z = -\nu_1^{(\alpha)} \text{ for } x = 0; \quad l_2 z = \nu_2^{(\alpha)} \text{ for } x = 1; \quad z = \varphi(x), \quad z_t = \bar{\varphi}(x) \quad \text{for } t = 0 \quad (22)$$

with conditions (7), where

$$\sigma_s \leqslant M \text{ or } \sigma_s = \infty \text{ (boundary condition of first kind).} \quad (23)$$

THEOREM 9. *If* $\alpha_2 = 0$, $\alpha_1 \geqslant 0{\cdot}5$ (or $\alpha_3 = 0$, $\alpha_1 \geqslant 0{\cdot}5$) *and the conditions* $a_{\bar{t}\bar{t}} \leqslant M$, $|(\sigma_s)_{\bar{t}\bar{t}}| \leqslant M$, $s = 1, 2$, *are satisfied, we have for the solution of problem* (21)–(23), *given sufficiently small* $\tau \leqslant \tau_0$:

$$\| z \|_0 \leqslant \tilde{M} \| z \|_7 \leqslant M\{\| z^1 \|_7 + \| \psi(x, t) \|_{5*} + \| \psi_{\bar{t}}(x, t) \|_{5*} + \| \psi(x, 0) \|_5 +$$
$$+ \| \psi_t(x, 0) \|_5 + [\sum_{t'=2\tau}^{t'=t} \tau(\| \psi(x, t') \|_{5*} + \| \psi_{\bar{t}}(x, t') \|_{5*} + \| \psi_{\bar{t}\bar{t}}(x, t') \|_{5*})^2]^{\frac{1}{2}}\} +$$
$$+ \overline{M}(\sum_{t'=\tau}^{t'=t} \tau \| \psi(x, t') \|_5^2)^{\frac{1}{2}} \quad (24)$$

where $\overline{M} = 0$ *for* $b \equiv 0$.

To prove the theorem, we write z as the sum $z = v + w$, where w is the solution of problem (1.1)–(1.4) for $d \equiv 0$, and make use of Lemma 6 and Theorem 8.

So far we have actually considered a seven-point difference equation of the form

$$\left(\begin{array}{ccc|c} * & * & * & \alpha_1 \\ & * & & \alpha_2 = 0 \\ * & * & * & \alpha_3 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{ccc|c} * & * & * & \alpha_1 \\ * & * & * & \alpha_2 \\ & * & & \alpha_3 = 0 \end{array}\right).$$

An *a priori* estimate will be obtained in the next section for the solution of the nine-point difference equation

$$\left(\begin{array}{ccc|c} * & * & * & 0{\cdot}25 \\ * & * & * & 0{\cdot}5 \\ * & * & * & 0{\cdot}25 \end{array}\right).$$

4. *An a priori estimate for the nine-point difference equation.* We consider the following problem:

$$\left.\begin{array}{l} \mathcal{P}_1 z = \psi^{(\alpha)} \text{ on } \Omega_2(\alpha_1 = \alpha_3 = 0{\cdot}25, \quad \alpha_2 = 0{\cdot}5), \\ z_0 = z_N = 0; \quad z(x, 0) = \varphi(x), \quad z_t(x, 0) = \bar{\varphi}(x), \end{array}\right\} \quad (25)$$

$$0 < M_0 \leqslant a \leqslant M_1, \quad 0 < M_0 \leqslant \rho \leqslant M_1, \quad |a_{\bar{t}}| \leqslant M, \quad |\rho_{\bar{t}}| \leqslant M,$$
$$|b| \leqslant M, \quad |d| \leqslant M, \quad |g| \leqslant M. \quad (26)$$

We rewrite the difference equation $P_2 z = \psi$ in the form

$$\sqrt{\rho}\, z_{\bar{t}} - 0{\cdot}25\, R\tau = \sqrt{\rho}\, \check{z}_{t} + 0{\cdot}25\, \tau \check{R} \sqrt{\frac{\check{\rho}}{\rho}} + \tau \Psi^{(\alpha)} / \sqrt{\rho}, \tag{27}$$

where

$$R = \frac{1}{\sqrt{\rho}} [(a z_{\bar{x}})_x + (\check{a} \check{z}_{\bar{x}})_x], \qquad \Psi = Q(z) + \psi. \tag{28}$$

On squaring (27), multiplying by h and summing over the net $\{x = h,\ 2h, \dots, (N-1)h\}$, we find

$$(\rho, z_{\bar{t}}^2) + 0{\cdot}5\, I + \frac{1}{16}\tau^2 \| R \|_2^2 = (\rho, z_{\bar{t}}^2) + 0{\cdot}5\, \check{\check{I}} + \frac{1}{16}\tau^2 \left\| \check{R} \sqrt{\frac{\check{\rho}}{\rho}} \right\|_2^2 +$$

$$+ \tau^2 \left\| \frac{1}{\sqrt{\rho}} \Psi \right\|_2^2 + 0{\cdot}5\, \tau^2 \left(\frac{1}{\rho} \Psi^{(\alpha)},\ \check{R} \sqrt{\check{\rho}} \right) + 2\tau (\Psi^{(\alpha)}, z_{\bar{t}}) +$$

$$+ 0{\cdot}5\, \tau (a_{\bar{t}}, z_{\bar{x}} \check{z}_{\bar{x}}] - 0{\cdot}5\, \tau (\check{a}_{\bar{t}}, \check{z}_{\bar{x}} \check{\check{z}}_{\bar{x}}], \tag{29}$$

where

$$I = (a,\ z_{\bar{x}}^2\,]. \tag{30}$$

For future transformations, we make use of (26), together with the inequalities

$$\tau \,|\, (a_{\bar{t}},\ z_{\bar{x}} \check{z}_{\bar{x}}] \,| \leqslant M \tau (I + \check{I}), \qquad \tau^2 \left| \left(\frac{1}{\rho} \Psi, \check{R} \sqrt{\check{\rho}} \right) \right| \leqslant M (\tau^3 \| \check{R} \|_2^2 + \tau \| \Psi \|_2^2),$$

$$\tau \| \Psi \|_2^2 \leqslant M \tau \{ I + \check{I} + \check{\check{I}} + \| \psi \|_2^2 + (\rho, z_{\bar{t}}^2) + (\check{\rho}, \check{z}_{\bar{t}}^2) \}.$$

As a result, we obtain for sufficiently small $\tau \leqslant \tau_0$ the following energy inequality:

$$E \leqslant (1 + M\tau)(\check{E} + M\tau \| \psi \|_2^2), \tag{31}$$

where

$$E = (\rho, z_{\bar{t}}^2) + 0{\cdot}5 (I + \check{I}) + \frac{1}{16} \tau^2 \| R \|_2^2. \tag{32}$$

An immediate consequence of (31) is

LEMMA 8. *The integral inequality*

$$E(t) \leqslant M \left(E(\tau) + \sum_{t'=2\tau}^{t'=t} \tau \| \psi(x, t') \|_2^2 \right) \quad \text{for } \tau \leqslant \tau_0, \tag{33}$$

holds for the solution of problem (25), (26), *where*

$$E(\tau) = (\rho(x, \tau), \bar{\varphi}^2) + 0{\cdot}5 \{ (a(x, 0), \varphi_{\bar{x}}^2] + (a(x, \tau), (\varphi_{\bar{x}} + \tau \bar{\varphi}_{\bar{x}})^2] \} +$$

$$+ \frac{\tau^2}{16} \| (a(x, 0)\, \varphi_{\bar{x}})_x + (a(x, \tau)\, \varphi_{\bar{x}})_x + \tau (a(x, \tau)\, \bar{\varphi}_{\bar{x}})_x \|_2^2. \tag{34}$$

It is easily seen that we can write, by (26),

$$E(\tau) \leqslant M \{ \| \bar{x} \|_2^2 + (\| \varphi_x \|_2^2 + \tau^2 \| \bar{\varphi}_{\bar{x}} \|_2^2)(1 + \gamma) \}$$

or

$$E(\tau) \leqslant M (\| \bar{\varphi} \|_2^2 + \| \varphi_{\bar{x}} \|_2^2)(1 + \gamma).$$

We have thus proved

THEOREM 10. *Given sufficiently small $\tau \geqslant \tau_0$, the solution of problem* (25), (26) *satisfies*

$$\| z \|_0 \leqslant \| z \|_7 \leqslant M \{E(\tau) + [\sum_{t'=2\tau}^{t'=t} \tau \| \psi(x, t') \|_2^2]^{\frac{1}{2}}\} \tag{35}$$

or

$$\| z \|_0 \leqslant \| z \|_7 \leqslant M \{\| z^1 \|_7 (1+\gamma) + [\sum_{t'=2\tau}^{t'=t} \tau \| \psi(x, t') \|_2^2]^{\frac{1}{2}}\}, \tag{36}$$

where

$$\| z \|_7 = \| z \|_2 + \| \check{z} \|_2 + \| z_{\bar{x}} \|_2 + \| \check{z}_{\bar{x}} \|_2 + \| z_{\bar{t}} \|_2 .$$

An analogue of Theorem 9 holds for problem (25), (26).

5. *Some remarks*

1) If the initial data are zero ($\varphi = \bar{\varphi} = 0$), the requirement in Theorems 8 and 10 that a and ρ be bounded from above becomes unnecessary.

2) Estimate (8) was obtained in [9] for the particular case $\alpha_1 = 1$ ($\alpha_2 = \alpha_3 = 0$) with the subsidiary assumption that $|a_x| \leqslant M$ and $\rho = 1$ for the first boundary problem ($z_0 = z_N = 0$).

3) We saw in § 2 that *a priori* estimates of the same kind hold for difference, differential and differential-difference (Rothe's method and the straight-line method) equations. A similar situation is observed in the case of equations of the hyperbolic type. There is therefore no need to write out the *a priori* estimates for differential-difference equations.

4) If the difference net $\bar{\Omega}$ is non-uniform, the difference equation $P_2 z = \psi$ has to be written in the form

$$\rho \check{z}_{\bar{t}\,\tilde{t}} - (a z_{\bar{x}})_{\hat{x}}^{(\alpha)} - Q(z) = \psi .$$

All the estimates obtained in this section retain their force for this equation, if we everywhere changes all the norms $\|\psi\|$ to $\|\psi\|^*$.

5) Similar results are obtained for a system of equations

$$\boldsymbol{\rho} \mathbf{z}_{\bar{t}\,\tilde{t}} - (\mathbf{a} \mathbf{z}_{\alpha})_x^{(\alpha)} - \mathbf{Q}(\mathbf{z}) = \boldsymbol{\psi},$$

where $\mathbf{a} = (a_{ij})$, $\boldsymbol{\rho} = (\rho_{ij})$ are positive definite matrices $(a_{ij} = a_{ji})$, $\mathbf{z} = \{z^j\}$, $\psi = \{\psi^j\}$ are net vector functions.

§ 4. FOURTH ORDER EQUATIONS

1. *Difference equations.* We consider the following problem:

$$\mathcal{P}_3 z = \rho z_{\bar{t}\,\bar{t}} + (a z_{\bar{x}x})_{\bar{x}x}^{(\alpha)} + Q(z) = \psi \text{ on } \Omega_3, \tag{1}$$

$$z = z_{x,0}, \qquad z_N = z_{\bar{x},N}^{\;*} = 0, \tag{2}$$

$$z(x, 0) = \varphi(x), \qquad z_t(x, 0) = \bar{\varphi}(x), \tag{3}$$

where

$$Q(z) = ((b z_{\bar{x}})_x + c z_{\tilde{x}} + d z)^{(\alpha)} + g z_{\bar{t}} + \check{g} \check{z}_{\tilde{x}} ; \tag{4}$$

$$v^{(\alpha)} = \alpha_1 v + \alpha_2 \check{v} + \alpha_3 \check{\check{v}}, \quad \alpha_1 \geqslant 0, \quad \alpha_2 \geqslant 0, \quad \alpha_3 \geqslant 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 1.$$

By hypothesis, the coefficients of the problem satisfy the conditions:

$$0 < M \leqslant a,\ 0 < M \leqslant \rho,\ |b| \leqslant M,\ |c| \leqslant M,\ |d| \leqslant M,\ |a_{\bar{t}}| \leqslant M,\ |\rho_t| \leqslant M. \tag{5}$$

All the results obtained below for problem (1)–(5) retain their force if we take instead of (2) any of the pairs of boundary conditions:

1) $z = z_x = 0$ for $x = 0$; $z_{\bar{x}\bar{x}} = (a^{(-1)}z_{\bar{x}\bar{x}})_{\bar{x}} = 0$ for $x = 1$;
2) $z_{xx} = (a^{(+1)}z_{xx})_x = 0$ for $x = 0$; $z = 0,\ z_{\bar{x}} = 0$ for $x = 1$;
3) $z = z_x = 0$ for $x = 0$; $z_{\bar{x}} = 0,\ (a^{(-1)}z_{\bar{x}\bar{x}})_{\bar{x}} = 0$ for $x = 1$;
4) $z_x = (a^{(+1)}z_{xx})_x = 0$ for $x = 0$; $z = z_{\bar{x}} = 0$ for $x = 1$,

Equation (1) is the difference analogue of the fourth order linear parabolic equation

$$\mathcal{L}_3 u = \frac{\partial^2}{\partial x^2}\left(k(x,t)\frac{\partial^2 u}{\partial x^2}\right) + c(x,t)\frac{\partial^2 u}{\partial t^2} + \left\{\frac{\partial}{\partial x}\left(b_1\frac{\partial u}{\partial x}\right) + b_2\frac{\partial u}{\partial x} + b_3 u + b_4\frac{\partial u}{\partial t}\right\} = \psi. \tag{6}$$

2. *An integral inequality. A priori* estimates for the solution of problem (1)–(5) may be obtained with the aid of integral relationships by analogy with § 3.

We multiply (1) by $h\tau(z_{\bar{t}}+\beta\check{z}_{\bar{t}})$, sum over the net $\{x = 2h, 3h, \ldots, (N-2)h\}$ and make use of Green's formula (B. 26). By virtue of boundary conditions (2) the substitutions vanish, and we obtain an identity, which becomes with $\beta = 1$, $\alpha_2 = 0$:

$$\tau((\rho, z_{\bar{t}}^2))_{\bar{t}} + \alpha_1 I = \alpha_3 I + \tau((\Psi, z_{\bar{t}} + \check{z}_{\bar{t}})) + (\alpha_1 a - \alpha^3\check{a}, z_{\bar{x}x}\check{z}_{\bar{x}x}) - \tau((\rho_{\bar{t}}, \check{z}_{\bar{t}}^2)), \tag{7}$$

where

$$\Psi = Q(z) + \psi, \qquad I = (a, z_{\bar{x}x}^2). \tag{8}$$

On proceeding by analogy with Section 2, § 3, we obtain from this the integral inequality for sufficiently small $\tau \leqq \tau_0$:

$$E \leqslant (1+M\tau)(\check{E} + M\tau\,\|\Psi\|_2^2), \quad \|\Psi\|_2^2 = ((\Psi, \Psi)), \tag{9}$$

where

$$E = ((\rho, z_{\bar{t}}^2)) + 0{\cdot}5\,(I + \check{I}). \tag{10}$$

We have, by conditions (2) and (5):

$$\|z\|_0 \leqslant \frac{1}{4}\,\|z_{\bar{x}}\|_2 \leqslant \frac{1}{16}\,\|z_{\bar{x}x}\|_2 \leqslant MI, \tag{11}$$

$$\rho \leqslant (1+M\tau)\,\check{\rho}, \qquad \tau\,|((\rho_{\bar{t}}, \check{z}_{\bar{t}}^2))| \leqslant M\tau((\check{\rho}, \check{z}_{\bar{t}}^2)), \tag{12}$$

$$\|\Psi\|_2^2 \leqslant M\,\{((\rho, z_{\bar{t}}^2)) + ((\check{\rho}, \check{z}_{\bar{t}}^2)) + I + \check{I} + \|\psi\|_2^2\}. \tag{13}$$

On now returning to identity (8) and taking (12), (13) into account we find the required integral ("energy") inequality:

$$E \leqslant (1+M\tau)(\check{E} + M\tau\,\|\psi\|_2^2) \quad \text{for } \tau \leqslant \tau_0,\ \alpha_1 \geqslant 0{\cdot}5. \tag{14}$$

The same inequality is got with $\beta = 0, \alpha_3 = 0, \alpha_1 \geqslant 0{\cdot}5$. In this case

$$E = ((\rho, z_{\bar{t}}^2)) + I.$$

3. *A priori estimates*

THEOREM 11. *If $z(x, t)$ is a solution of problem* (1)–(5) *and* $\alpha_2 = 0$, $\alpha_1 \geqslant 0{\cdot}5$, we have for sufficiently small $\tau \leqslant \tau_0$:

$$\| z \|_0 \leqslant \| z \|_8 \leqslant M \Big\{ \| z^1 \|_8 + \Big[\sum_{t'=2\tau}^{t'=t} \tau \| \psi(x, t') \|_2^2 \Big]^{\frac{1}{2}} \Big\}, \quad z^1 = z(x, \tau),$$

where

$$\| z \|_8 = \| z \|_2 + \| \check{z} \|_2 + \| z_{\bar{x}} \|_2 + \| \check{z}_{\bar{x}} \|_2 + \| z_{\bar{x}x} \|_2 + \| \check{z}_{\bar{x}x} \|_2 + \| z_{\bar{t}} \|_2 .$$

If $\alpha_3 = 0$, this estimate retains its force with the condition that

$$\| z \|_8 = \| z \|_2 + \| z_{\bar{x}} \|_2 + \| z_{\bar{x}x} \|_2 + \| z_{\bar{t}} \|_2 .$$

To prove the theorem, we solve inequality (14) and at the same time take (11) and (5) into account.

Theorems analogous to Theorems 9 and 10 hold for our problem (1)–(5). We shall omit their statement, so as not to overload the discussion. Also, their proofs give rise to no new factors.

There is no difficulty about writing down corresponding estimates for differential and differential-difference equations.

We omit here the case of boundary conditions of a higher order of accuracy, since this substantially increases the complexity of the treatment.

Transition to a system of equations also presents no difficulty.

The method developed here enables estimates to be obtained (and hence convergence theorems to be proved in a class of discontinuous coefficients) for higher order equations.

A study of difference schemes for fourth order equations on a non-uniform net deserves attention.

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