THE STURM-LIOUVILLE DIFFERENCE PROBLEM*

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The solution of the Sturm-Liouville problem for the equation

\[ L^{(k,q)} u + \lambda r(x) u = 0, \quad 0 < x < 1, \quad L^{(k,q)} u = \frac{d}{dx} \left[ k(x) \frac{du}{dx} \right] - q(x) u(x) \quad (1) \]

by the method of finite differences has been the subject of a great many studies. The question of convergence and accuracy in the class of smooth coefficients for difference schemes of a particular form were considered in [1]–[3].

In this paper we employ the homogeneous difference schemes, studied in [4], for solving the Sturm–Liouville problem in the class of discontinuous coefficients \( Q^{(m)} \). The formulation of the problem and the characteristics of the initial family of difference schemes are given in § 1. In § 2 the convergence of the difference method is proved. In § 3, with the aid of \textit{a priori} estimates we establish the order of accuracy in \( Q^{(m,1)} \) of the solution of the difference problem when \( h \to 0 \). It is shown that the difference scheme

\[ L^{(k,q,\lambda r)} y = (ay_x)_x - dy + \lambda \varphi y, \]

where

\[ a = \left[ \int_{-1}^{0} \frac{ds}{k(x+sh)} \right]^{-1}, \quad d = \int_{-0.5}^{0.5} q(x+sh) \, ds, \quad \varphi = \int_{-0.5}^{0.5} r(x+sh) \, ds, \]

ensures the second order of accuracy in the class of discontinuous coefficients.

§ 1. DIFFERENCE BOUNDARY-VALUE PROBLEM

1. Sturm–Liouville problem. We shall consider the homogeneous differential equation (1) with homogeneous boundary conditions

\[ k(0) u'(0) + \sigma_1 u(0) = 0, \quad k(1) u'(1) - \sigma_2 u(1) = 0, \quad (2) \]

where \( \sigma_1 \) and \( \sigma_2 \) are some constants.

The Sturm–Liouville problem, or the eigenvalues problem, consists in finding those values of the parameter \( \lambda \) (eigenvalues) at which there exist non-trivial solutions (eigenfunctions) of the problem (1)–(2) and also in finding the eigenfunctions.

Henceforward we shall everywhere assume that the coefficients entering into equation (1) and the condition (2) satisfy the conditions

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\[ 0 < c_1 \leq k(x) \leq c_2, \quad 0 < c_3 \leq r(x) \leq c_4, \quad 0 \leq q(x) \leq c_5, \]
\[ \sigma_1 \geq 0, \quad \sigma_2 \geq 0, \quad \sigma_1 + \sigma_2 > 0. \]  
(3)

where \( c_j \) \((j = 1, \ldots, 5)\) are some constants.

If \( k(x) \) has a discontinuity of the first kind at the point \( x = \xi (0 < \xi < 1) \), then at this point the conditions of conjugation (continuity of \( u(x) \) and \( k(x) u'(x) \)) must be satisfied:

\[ [u] = 0, \quad [ku'] = 0 \quad \text{when} \quad x = \xi, \]
(4)

where

\[ [f] = f_1 - f_0, \quad f_1 = f(\xi^0), \quad f_0 = f(\xi^0). \]

The problem defined by condition (1)-(4) will henceforward be termed problem (I).

We shall use the following notations: \( Q^n [l_1, l_2] \) is a class of functions which are piecewise-continuous on the line segment \([l_1, l_2]\) together with their derivatives to the \( m \)th order inclusive; \( Q(\gamma)[l_1, l_2] \) \((0 < \gamma < 1)\) is a class of functions from \( Q^n [l_1, l_2] \), the \( m \)th derivatives of which satisfy on the intervals of their continuity the Hölder condition of the order \( \gamma \); \( C(\gamma)[l_1, l_2] \) is, as usual, a class of functions having an \( m \)th continuous derivative.

As we know, problem (I) is equivalent to the variation problem:

(1) In the class of piecewise-smooth comparison functions \( \varphi(x) \), satisfying the conditions

\[ H[\varphi] = \int_0^1 \varphi^2(x) r(x) \, dx = 1, \]

\[ k(0) \varphi'(0) - \sigma_1 \varphi(0) = 0, \quad k(1) \varphi'(1) + \sigma_2 \varphi(1) = 0, \]

find the minimum of the functional

\[ D[\varphi] = \int_0^1 k(x) (\varphi')^2 \, dx + \int_0^1 q(x) \varphi^2(x) \, dx + \sigma_1 \varphi^2(0) + \sigma_2 \varphi^2(1). \]
(5)

This minimum determines the first eigenvalue

\[ \lambda_1 = \min D[\varphi] = D[u_1] \]

(2) The remaining eigenvalues \( \lambda_n (n > 1) \) are found as the minimum of functional (5) in the class of piecewise-smooth comparison functions \( \varphi(x) \) satisfying the additional conditions

\[ H[\varphi] = 1, \quad H[\varphi, u_m] = \int_0^1 \varphi(x) u_m(x) r(x) \, dx = 0 \quad \text{when} \quad m < n, \]

\[ k(0) \varphi'(0) - \sigma_1 \varphi(0) = 0, \quad k(1) \varphi'(1) + \sigma_2 \varphi(1) = 0, \]

where \( u_m(x) \) is the eigenfunction of the number \( m \). This minimum determines the \( n \)th eigenvalue

\[ \lambda_n = \min D[\varphi] = D[u_n] \]

and is reached at the \( n \)th eigenfunction \( u_n(x) \).
The Sturm-Liouville problem (I) for piecewise-continuous coefficients \( k, q, r \in C(0) \) has the denumerable set of eigenvalues \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots \), to which there correspond the eigenfunctions \( u_1(x), u_2(x), \ldots, u_n(x), \ldots \) (see [5]).

We shall indicate certain known [5] properties of eigenfunctions and eigenvalues.

1. To each eigenvalue there corresponds only one eigenfunction.

Let us assume that to \( \lambda_n \) there correspond two eigenfunctions \( u_n(x) \) and \( u_n^*(x) \). Then their linear combination \( u_n(x) = u_n(0)u_n^*(x) - u_n^*(0)u_n(x) \) satisfies the condition \( u_n(0) = 0 \) and is also the eigenfunction corresponding to \( \lambda_n \). From the condition \( k(0)\ddot{u}(0) - \sigma_1\dot{u}(0) = 0 \) for \( \sigma_1 \neq 0 \) we obtain \( \ddot{u}(0) = 0 \). It follows from this that \( u_n(x) \equiv 0 \). In the case of the condition of the first kind (\( \sigma_1 = 0 \)) one can select \( u_n(x) = u_n^*(0)u_n(x) - u_n(0)u_n^*(x) \). We can therefore write \( \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots \).

2. Eigenfunctions \( \{u_n(x)\} \) form an orthogonal system normalized with weight \( r(x) \):

\[
H[u_n, u_m] = 0 \quad \text{when} \quad m \neq n, \quad H[u_n] = 1.
\]

3. All eigenvalues are positive: \( \lambda_n > 0, \ n = 1, 2, \ldots \).

4. Eigenvalues \( \lambda_n \to \infty \) when \( n \to \infty \), or, more accurately,

\[
c_6n^2 \leq \lambda_n \leq c_7n^2,
\]

where \( c_6 \) and \( c_7 \) are positive constants independent of the number \( n \) and dependent solely on \( c_j(j = 1, \ldots, 5) \).

5. Eigenfunctions and their first derivatives are bounded — more accurately,

\[
|u_n(x)| < c_8, \quad |u_n'(x)| < c_9\sqrt{\lambda_n} \leq c_9n,
\]

where \( c_8 \) and \( c_9 \) are positive constants dependent solely on \( c_j(j = 1, \ldots, 5) \).

For the case \( k, q, r \in C(0) \) proof of (7) is given in [5]. This proof may also be transferred, with certain changes, to the case \( k, q, r \in C(0) \).

Let us introduce the new variable \( t = \int_0^x r(x) \, dx \). Then equation (1) takes the form

\[
\frac{d}{dt} \left[ -k(t) \frac{d\ddot{u}}{dt} \right] - \ddot{u} + \lambda \ddot{u} = 0, \quad 0 < t < l, \tag{1'}
\]

where

\[
\ddot{u}(t) = k(x)r(x), \quad \ddot{u}(t) = q(x)/r(x), \quad \ddot{u}(t) = u(x), \quad l = \int_0^1 r(x) \, dx.
\]

We multiply equation (1') by \( \ddot{u}'(t) \) and integrate from 0 to \( t \):

\[
\int_0^t \ddot{u}(0)\ddot{u}'(0) + \lambda \ddot{u}(0) = \int_0^t \ddot{u} + \lambda \ddot{u} \, dt - 2 \int_0^t \ddot{u} \ddot{u}' \, dt. \tag{8}
\]

We integrate once more with respect to \( t \) from 0 to \( l \), and bear in mind that
\[
\int_0^l \tilde{u}^2(t) \, dt = 1, \quad \int_0^l \tilde{k}(t) \tilde{u}'(t) \, dt \leq \lambda,
\]
\[
2 \int_0^l \left| \int_0^l \tilde{q} \tilde{u}' \, dt \right| \leq 2l \cdot c_5 \left( \int_0^l \tilde{u}^2 \, dt \right)^{\frac{1}{2}} \leq \frac{2lc_5}{c_1c_3} \sqrt{\lambda} \leq c_{10} \sqrt{\lambda}.
\]

As a result we will have
\[
\tilde{k}(0) \tilde{u}'(0) + \lambda \tilde{u}^2(0) \leq \frac{2\lambda}{l} + c_{10} \sqrt{\lambda}.
\]  

Returning now to (8) and bearing in mind inequalities (9) and (10) we obtain
\[
\tilde{k}(t) \tilde{u}'(t) + \lambda \tilde{u}^2(t) \leq \frac{2\lambda}{l} + c_{10} \sqrt{\lambda}.
\]

From this, and from (6) there follow estimates (7).

Problem (1) is equivalent to an integral equation with synthetizable kernel:
\[
u(x) = \lambda \int_0^1 G(x, \xi) r(\xi) u(\xi) \, d\xi,
\]
where \( G(x, \xi) \) is Green's function for the operator \( L^{(k, d)} \) with boundary conditions (2), and also to the integral equation
\[
u(x) = \int_0^1 G_0(x, \xi) (\lambda r(\xi) - q(\xi)) u(\xi) \, d\xi,
\]
where \( G_0(x, \xi) \) is Green's function for the operator \( L^{(k, d)} = (ku')' \) with boundary conditions (2).

2. Notations. Let us consider on the line segment \([0, 1]\) the difference mesh
\[\omega_h = \{x_0 = 0, \ldots, x_i = i \cdot h, \ldots, x_N = N \cdot h = 1\}.
\]

The mesh function \(y_i\), defined by \(\omega_h\), will henceforward be designated by \(y\) or \(y(x)\).

When this will not give rise to misunderstanding. We shall also write
\[y^{(+1)} = y_{i+1}, \quad y^{(-1)} = y_{i-1}, \quad y_x = (y^{(-1)} - y^{(+1)})/h, \quad y_x = (y^{(-1)} - y^{(+1)})/h.
\]

Let \(v\) be some mesh function defined on \(\omega_h\). We shall introduce the notations
\[\langle y, v \rangle = \sum_{i=1}^{N-1} y_i v_i h, \quad \langle y, v \rangle = \sum_{i=0}^{N-1} y_i v_i h, \quad \langle y, v \rangle = \sum_{i=0}^{N} y_i v_i h, \quad \langle y, v \rangle = \sum_{i=0}^{N} y_i v_i h.
\]

We shall use the following norms:
\[\|v\|_0 = \max_{\omega_h} |v_i|, \quad \|v\|_o = \|[v\delta, 1]\|_0 \quad (\sigma = 1, 2),
\]
\[\|v\|_3 = \sum_{i=1}^{N-1} h \left| \sum_{k=1}^{l} h v_k \right|,\]
\[\|v\|_4 = \|v\|_3 + \langle (v, 1) \rangle + \langle v_0 \rangle h + \langle v_N \rangle h.
\]
It may be found that the function \( v \) is defined not on the whole mesh, but on part of it. For example, the function \( y_x \) is defined at the points \( x_i \) for \( i = 1, 2, \ldots, N \), and the function \( y_x \) for \( i = 0, 1, 2, \ldots, N-1 \). In this case

\[
\|y_x\|_0 = \max_{0 \leq i < N} |y_x(i)|, \quad \|y_x\|_\sigma = (|y_x|^\sigma, 1)^{1/\sigma}, \quad \sigma = 1, 2.
\]

\[
\|y_x\|_0 = \max_{0 \leq i < N} |y_x(i)|, \quad \|y_x\|_\sigma = (|y_x|^\sigma, 1)^{1/\sigma}.
\]

We shall also write

\[
[y, \psi] = (y, \psi) + y_0 \bar{v}_1 + y_N \bar{v}_2, \quad (\text{15})
\]

where \( \psi \) is a function defined at the points \( x_i \), \( i = 1, 2, \ldots, N-1 \), if we formally put \( h\psi_0 = \bar{v}_1, h\psi_N = \bar{v}_2 \).

The difference operator \( (a_i+1(y_{i+1}-y_i) - a_i(y_i-y_{i-1})/h^2 \) with the aid of the notations introduced, will be written in the form \((ay_x)_x\).

We shall need henceforward:

(1) the formula of summation by parts:

\[
[y, v_x] = -(v, y_x) + (yv)_x - (yv)_0; \quad (\text{16})
\]

(2) Green's difference formulae

\[
((ay_x)_x, v) = -(a, y_x v_x) + (ay_x v)_x - (a^{(i+1)} y_x v)_0, \quad (\text{17})
\]

\[
((ay_x)_x, v) = ((av_x)_x, y) + a_N (y_x v - yv_x)_x - a_1 (y_x v - yv_x)_0. (\text{18})
\]

We shall not deal with the derivation of these formulae, since it is elementary. We shall designate by \( \rho(h) \) any expression, uniformly converging to zero when \( h \to 0 \).

All constants independent of \( h \) will be designated by the letter \( M \), without, as a rule, indicating their structure or their relationship to other constants.

3. Difference schemes. In deriving a difference scheme for the solution of problem (I) we shall use the results of [4]. We shall introduce the notations

\[
L^{(k, a, \lambda r)} u = (ku)^\prime - qu + \lambda ru, \quad L^{(k, a, \lambda r)} y = (ay_x)_x - dy + \lambda cy.
\]

We shall assume that \( L^{(k, a, \lambda r)} \) is a homogeneous conservative scheme of the standard type [4]. Its coefficients \( a, d \) and \( \rho \) are expressed with the aid of the standard functionals

\[
A^h[f(s)], \quad f \in Q^{(0)} [-1; 0]; \quad D^h[f(s)], \quad f \in Q^{(0)} [-0.5; 0.5];
\]

\[
R^h[f(s)], \quad f \in Q^{(0)} [-0.5; 0.5],
\]

defined in the class \( Q^{(0)}(f \in Q^{(0)}) \), in terms of the coefficients \( k, q, r \) of the differential equation (I). Each coefficient of the scheme depends on only one coefficient of the differential equation (standard-type scheme):

\[
a = A^h[k(x+sh)], \quad -1 < s < 0; \quad d = D^h[q(x+sh)], \quad -0.5 < s < 0.5;
\]

\[
\rho = R^h[r(x+\rho h)], \quad -0.5 < s < 0.5.
\]

(The dependence of the coefficients \( a, d \) and \( \rho \) on \( h \) is not explicitly indicated).
As the initial family of schemes we shall consider conservative schemes, since, as shown in [4], non-conservative schemes

\[ \bar{L}_h^{(k, q)} y = (b y_x - a y_x)h - dy \]

do not give convergence in $Q^{(m)}$. In the class of smooth coefficients the scheme $\bar{L}_h^{(k, q)}$ can be transformed into the conservative scheme

\[ L_h^{(k, q)} y_i = \mu_i \bar{L}_h^{(k, q)} y_i = (\bar{a} y_y)_x, \quad \bar{d}_i y_i, \]

where

\[ \mu_i = \prod_{k=1}^{i-1} (b_k/\alpha_k + 1), \quad \bar{a}_i = a_i \mu_i, \quad \bar{d}_i = d_i \mu_i \]

(see [4]).

If the scheme $\bar{L}_h^{(k, q)}$ has an $m$th ($m = 1, 2$) order of approximation and $k(x) \in C^{(m+1)}$, then $\mu_i = 1 + O(h^m)$. From this it follows that the results obtained henceforth for $k(x) \in C^{(m+1)}$, $q, r \in C^{(m)}$ will also be transferable to non-conservative schemes.

We note that the authors of [1]–[3] only studied discrete schemes of a particular form (e.g., $a = k(x - h), a = k(x - 0.5h), d = q, \rho = r(x)$), whilst in addition discrete schemes of the fourth order in $C^{(m)}$ were studied in [3].

The difference scheme $L_h^{(k, q, \lambda_r)}$ is characterized by an approximation error in the class of differentiable coefficients

\[ \psi = L_h^{(k, q, \lambda_r)} u - L_h^{(k, q, \lambda_r)} u, \]

where $u$ is any function differentiable a sufficient number of times.

In studying convergence and accuracy, we shall be dealing with the approximation error on the solution $u(x)$ of equation (1).

In [5] the concept of the rank of standard functionals was introduced.

If all the standard functionals have a rank $m(m = 0, 1, 2)$, we say that $L_h^{(k, q, \lambda_r)}$ is a scheme of $m$th rank. The requirement of definiteness of the order of approximation leads to certain conditions (which will not be written out here) to be satisfied by the standard functionals and their differentials with respect to $h$ and with respect to the functional argument. We shall indicate two properties of standard functionals which are fulfilled for schemes of all ranks, beginning with a scheme of zero rank:

(a) $A[h] = D[h] = R[h] = 1$ (normalization condition)

(b) the functionals $A[f], D[f]$ and $R[f]$ are non-decreasing: $A[f] \geq A[f']$ if $f' \geq f$, and so on.

Moreover, we assume that $D[f]$ and $R[f]$ are linear functionals.

From this, and from (3) in particular it follows that the coefficients $a, d$ and $\rho$ satisfy the conditions

\[ 0 < c_1 \leq a \leq c_2, \quad 0 < c_3 \leq \rho \leq c_4, \quad 0 \leq d \leq c_5, \]

where $c_j(j = 1, \ldots, 5)$ are constants entering into condition (3).

As the initial class of difference schemes $L_h^{(k, q, \lambda_r)}$ we shall consider in § 2 schemes of zero rank.
In § 3 we shall also examine the family of difference schemes of the second rank satisfying the conditions of the second order of approximation (see [4]). This family belongs to the initial class, and we shall call it the initial family of schemes of the second order.

If the standard functionals are canonical, i.e., do not depend on the parameter \( h \), the difference scheme is called canonical.

Two schemes are termed equivalent with respect to the order of approximation (accuracy) if they have an identical order of approximation (accuracy). It is not difficult to see that any scheme of \( m \)th rank is equivalent to its canonical part [4], i.e., to a scheme with canonical standard functionals \( A^{(0)} \), \( D^{(0)} \) and \( R^{(0)} \) which are the main terms in the expansion of \( A^h \), \( D^h \) and \( R^h \) with respect to \( h \).

It follows from this that the entire exposition may be made for canonical schemes.

We shall give more detailed characteristics of the properties of the canonical standard functionals \( A[\cdot], D[\cdot] \) and \( R[\cdot] \).

The linear canonical functionals \( D[\cdot] \) and \( R[\cdot] \), satisfying conditions (a) and (b) have any rank, as high as one wishes.

The canonical functional \( A[\cdot] \), which is generally speaking non-linear, has \( m \)th rank if, apart from conditions (a) and (b), it satisfies two further requirements:

(c) \( A[\cdot] \) is an homogeneous functional of the first degree, i.e.,

\[
A[cf] = cA[\cdot],
\]

where \( c \) is any positive number:

(d) \( A[\cdot] \) has an \( m \)th differential, so that in particular we may write

\[
\begin{align*}
A[1 + \delta \cdot f] &= 1 + \rho(\delta) \quad \text{(if } |f| \ll M) \quad \text{when } m = 0, \\
A[1 + \delta \cdot f] &= 1 + \delta A_1[f] + \delta^2 \rho(\delta) \quad \text{when } m = 1, \\
A[1 + \delta \cdot f] &= 1 + \delta A_1[f] + \delta^2 A_2[f] + \delta^3 \rho(\delta) \quad \text{when } m = 2,
\end{align*}
\]

where \( \rho(\delta) \to 0 \) when \( \delta \to 0 \), \( A_1[\cdot] \) is a linear functional and \( A_2[\cdot] \) a quadratic functional.

The necessary conditions for the \( m \)th order of approximation of the scheme \( L^{(k,q,r)}_h = (k,q,r \in C^{(m)}[0,1]) \) have the form \( a = k(\bar{x}) + O(h^m), \quad d = q(\bar{x}) + O(h^m), \quad \rho = r(\bar{x}) + O(h^m), \quad \bar{x} = x - 0.5h, \quad (20) \)

Any canonical scheme of the first rank satisfies these conditions when \( m = 1 \).

A canonical scheme of second rank satisfies the conditions (20) when \( m = 2 \), if

\[
\begin{align*}
A_1[s] &= -0.5, \quad D[s] = 0, \quad R[s] = 0.5. \quad \text{(21)}
\end{align*}
\]

It is easy to show that for any symmetrical scheme of the second rank conditions (21) are satisfied.

4. Difference boundary-value problem. Before formulating the difference analogue of problem (1) we have to formulate difference boundary conditions corresponding to conditions (2).
We shall consider the boundary operator
\[ l^{(1)}u = k(0) u'(0) - \sigma_1 u(0) \]
and its difference analogue
\[ \tilde{l}^{(1)}_h y = a_1 y_{x,0} - \sigma_1 y_0. \]

It is easy to show that the operator \( \tilde{l}^{(1)}_h \) has only a first order of approximation. In fact, let \( L_h^{(2)} \) be a scheme of second rank, satisfying the conditions (21) for the second order of approximation, so that \( a = k(x - 0.5h) + O(h^2) \) for \( k \in C^2 \).

We shall then have \( a_1 = k(0) + 0.5hk'(0) + O(h^2), \ u_x,0 = u'(0) + 0.5hu''(0) + O(h^2), \)
\[ \tilde{l}^{(1)}_h u - l^{(1)} u = 0.5h(ku')'|_{x=0} + O(h^2) = 0.5h(q(0) - \lambda r(0)) u(0) + O(h^2), \]
where \( u(x) \) is a solution of equation (1).

It follows from this that the difference operator
\[ l_h^{(1)} y = a_1 y_{x,0} - \sigma_1 y + 0.5h\lambda r(0) y_0, \quad \text{where} \quad \sigma_1 = \sigma_1 + 0.5hq(0), \]
has a second order of approximation in the class of solutions of equation (1).

The same property is possessed by the operator
\[ l_h^{(2)} y = a_N y_{x,N} - \sigma_2 y_N - 0.5h\lambda r(1) y_N, \quad \text{where} \quad \sigma_2 = \sigma_2 + 0.5hq(1), \]
which corresponds to the operator
\[ l^{(2)} u = k(1) u'(1) + \sigma_2 u(1). \]

The Sturm-Liouville difference problem will be formulated as follows. We are required to find those values of the parameter \( \lambda^h \) (eigenvalues) to which there correspond non-trivial solutions (eigenfunctions) of the homogeneous equation
\[ (ay^2)_x - dy + \lambda \varphi y = 0 \]
with homogeneous boundary conditions
\[ a_1 y_{x,0} - \sigma_1 y_0 + h\lambda \varphi_0 y_0 = 0, \quad a_N y_{x,N} - \sigma_2 y_N - h\lambda \varphi_N y_N = 0, \]
where \( \varphi_0 = 0.5r(0), \ \varphi_N = 0.5r(1), \ \sigma_1 = \sigma_1 + 0.5hq(0), \ \sigma_2 = \sigma_2 + 0.5hq(1), \)
and also to find these non-trivial solutions (eigenfunctions).

The coefficients of the problem satisfy the conditions
\[ 0 < c_1 < a < c_2, \quad 0 < c_3 < \varphi < c_4, \quad 0 < d < c_5, \quad \sigma_1 \geq 0, \ \sigma_2 \geq 0, \]
\[ \sigma_1 + \sigma_2 > 0. \quad (24) \]

In the neighbourhood of discontinuity of the coefficient \( k \), the conditions of continuity, like conditions (4), are absent, since we are considering homogeneous schemes, which do not envisage the explicit isolation of the points of discontinuity of the coefficients of equation (1) (schemes of through-computation).

Henceforth the difference boundary-value problem (22)-(24) will be termed problem (II).
5. **Difference variational problem.** Let \( y \) and \( \lambda^h \) be the eigenfunction and the corresponding eigenvalue of problem (II). Putting \( v = y \) in formula (17) and using conditions (23) we find

\[
\lambda^h = D_N[y] / H_N[y],
\]

where

\[
D_N[y] = [a, y^2_0 + \overline{\sigma_1} y^2_0 + \overline{\sigma_2} y^2_N], \quad (y^2_x = (y_x)^2),
\]

\[
H_N[y] = [p, y^2], \quad \rho_0 = 0.5r(0), \quad \rho_N = 0.5r(1).
\]

Using Green's formula, it is easy to see that difference boundary-value problem (II) is equivalent to the following variational problem:

1. find the minimum of the functional \( D_N[\varphi] \) in the class of mesh functions \( \varphi \), satisfying the conditions

\[
a_1 \varphi_{x,0} - \overline{\sigma_1} \varphi_0 + hD_N[\varphi] \rho_0 \varphi_0 = 0, \quad a_N \varphi_{x,N} + \overline{\sigma_2} \varphi_N - hD_N[\varphi] \rho_N \varphi_N = 0,
\]

\[
H_N[\varphi] = 1;
\]

and

\[
\lambda^h = D_N[y_1] = \min D_N[\varphi]
\]

is the least eigenvalue and \( y_1 \) the corresponding eigenfunction of problem (II);

2. find the minimum of \( D_N[\varphi] \) in the class of comparison functions satisfying the conditions

\[
H_N[\varphi] = 1, \quad H_N[\varphi, y_m] = [p, y_m] = 0 \quad \text{for} \ m = 1, 2, ..., n-1,
\]

\[
a_1 \varphi_{x,0} - \overline{\sigma_1} \varphi_0 + hD_N[\varphi] \rho_0 \varphi_0 = 0, \quad a_N \varphi_{x,N} + \overline{\sigma_2} \varphi_N - hD_N[\varphi] \rho_N \varphi_N = 0,
\]

where \( y \) is the eigenfunction of the number \( m \); moreover

\[
\min D_N[\varphi] = D_N[y_m] = \lambda^h
\]

is the \( n \)th eigenvalue and \( y_m(x) \) its eigenfunction.

6. **Integral relation.** Let us examine Green's difference function \( G^h(x, \xi) \), which is definable from the conditions:

\[
(aG^h_x)_x - dG^h = - \frac{\delta(x, \xi)}{h}, \quad \text{where} \ \delta(x, \xi) = \begin{cases} 1, & x = \xi, \\ 0, & x \neq \xi, \end{cases}
\]

\[
a_1 G^h_{x,0} - \overline{\sigma_1} G^h_0 = 0, \quad a_N G^h_{x,N} + \overline{\sigma_2} G^h_N = 0.
\]

Introducing the functions \( \alpha^h \) and \( \beta^h \) which are solutions of the homogeneous equation

\[
(a\alpha^h_x)_x - d\alpha^h = 0,
\]

satisfying the initial conditions

\[
a_1 \alpha^h_{x,0} = 1, \quad a_1 \alpha^h_{x,0} - \overline{\sigma_1} \alpha^h_0 = 0; \quad a_N \beta^h_{x,N} = -1, \quad a_N \beta^h_{x,N} + \overline{\sigma_2} \beta^h_N = 0,
\]

we represent Green's function, by analogy with [4], in the form

\[
G^h(x, \xi) = \begin{cases} \frac{1}{\Delta} \alpha^h(x) \beta^h(\xi), & x < \xi, \\ \frac{1}{\Delta} \alpha^h(\xi) \beta^h(x), & x > \xi, \end{cases}
\]
\[ \Delta = \alpha^h + \frac{1}{\sigma_2} [1 + (d, \alpha^h)] = \beta^h + \frac{1}{\sigma_1} [1 + (d, \beta^h)] = \text{const}. \]

From this there immediately follows the symmetry of Green's function
\[ G^h(x, \xi) = G^h(\xi, x). \]

By analogy with [4] it is easy to show that \( G^h \) and its difference derivatives \( G^h_x \) and \( G^h_\xi \) are bounded:
\[ 0 \leq G^h < M_1, \quad \| G^h_x \|_0 \leq M_2, \| G^h_\xi \|_0 \leq M_2, \]

where \( M_1 \) and \( M_2 \) are positive constants independent of \( h \).

If \( d = 0 \), then \( \alpha^h \) and \( \beta^h \) have the form
\[ \alpha^h(x) = \frac{1}{\sigma_1} + \sum_{x = x_0}^{x = x_0 + h} a(x'), \quad \beta^h(x) = \frac{1}{\sigma_2} + \sum_{x = x_0}^{x = x_0 + h} \frac{h}{a(x')} - \frac{1}{\sigma_2}, \]
\[ \Delta = \frac{1}{\sigma_1} + \frac{1}{\sigma_2} + (1, 1/\sigma). \]

If, moreover, \( \sigma_1 = 0 \), then \( \alpha^h(x) = 1, \Delta = 1 \)

From now on we shall require

**Lemma 1.** Let \( G_0^h(x, \xi) \) be Green's function of the difference operator \((ay_{\xi\xi})_x\) with boundary conditions (23), and \( G_0(x, \xi) \) Green's function of the differential operator \((ku')_x\) with conditions (2). If \( k(x) \in Q^0 \) and \( \| a - k \|_1 = \rho(h) \), then \( \| G^h_0 - G_0 \|_0 = \rho(h) \), where \( \rho(h) \to 0 \) when \( h \to 0 \).

In fact, noting that
\[ \| \alpha^h - \alpha^0 \|_0 = \rho(h), \quad \| \beta^h - \beta^0 \|_0 = \rho(h), \]
where
\[ \alpha^0(x) = \frac{1}{\sigma_1} + \int_0^x \frac{dt}{k(t)}, \quad \beta^0(x) = \frac{1}{\sigma_2} + \int_0^x \frac{dt}{k(t)}, \]
we find
\[ \| G_0^h - G_0 \|_0 \leq M \| a - k \|_1 = \rho(h), \]

since \( G_0 \) is defined by the same formula (23), in which \( \alpha^h \) and \( \beta^h \) are replaced by the functions \( \alpha^0 \) and \( \beta^0 \).

Using the second Green's formula (18) it is easy to see that the Sturm–Liouville difference problem is equivalent to the difference analogue of the integral equation
\[ y = \lambda^h [G^h, y], \tag{29} \]

which, with the aid of the substitution
\[ \varphi(x) = \sqrt{\rho(x)} y(x), \quad K^h(x, \xi) = \sqrt{\rho(x)} \rho(\xi) G^h(x, \xi) \]
is reduced to the equation
\[ \varphi = \lambda^h [K^h, \varphi], \tag{30} \]

with symmetrical kernel \( K^h(x, \xi) \).
If we use Green’s function $G_0$ of the operator $L_h^{(k)}$, instead of (30) we obtain the equation

$$y = [G_0, (\lambda \rho - d)] y,$$

(31)

where $d_0 = 0.5q (0), \quad d_N = 0.5q (1), \quad \rho_0 = 0.5r (0), \quad \rho_N = 0.5r (1)$.

7. Properties of eigenfunctions and eigenvalues. The Sturm–Liouville difference problem (II) is a purely algebraic problem. It is therefore not difficult to prove the following propositions:

1. there exist $N$ real eigenvalues $\lambda_1^h, \lambda_2^h, \ldots, \lambda_N^h$, to which there correspond the eigenfunctions $y_1, y_2, \ldots, y_N$;

2. to each eigenvalue there corresponds only one eigenfunction (this is proved in the same way as for problem (I) in Section 1), so that we can write $\lambda_1^h < \lambda_2^h \ldots \lambda_N^h$;

3. all eigenvalues are positive (this follows from (25));

4. eigenfunctions form an orthogonal system normalized with weight $\rho$

$$H_n[y_m, y_n] = [\rho y_m, y_n] = \begin{cases} 0, & m \neq n, \\ 1, & m = n; \end{cases}$$

(32)

where $M_1$ and $M_2$ are positive constants independent of both $h$ and $n$.

LEMMA 2. Let $y_n, \lambda_n^h$ be the $n$-th normalized eigenfunction and the $n$-th eigenvalue of problem (II). Then the functions $y_n$ and $(y_n)_x$ are uniformly bounded:

$$\|y_n\|_0 \leq M_1 n^{-\frac{1}{2}}, \quad \|(y_n)_x\|_0 \leq M_2 n^{-\frac{3}{2}}$$

(33)

where $M_1$ and $M_2$ are constants independent of both $h$ and $n$.

Let $x$ and $x'$ be any two points of the mesh $\omega_h$. Let us examine the two obvious identities:

$$y^2 (x) - y^2 (x') = \sum_{s=x-h}^{s=x} (y^2 (s))_x \cdot h = \sum_{s=x-h}^{s=x} [y(s) + y(s-h)] y_x (s) \cdot h,$$

(34)

\[ (a (x) y_x (x))^2 - (a (x') y_x (x'))^2 = \sum_{s=x-h}^{s=x} h \left[ (a (s) y_x (s))^2 - (a (s+h) y_x (s))^2 \right] \]

\[ = \sum_{s=x-h}^{s=x} (a (s) y_x (s)) \cdot [a (s) y_x (s) + a (s+h) y_x (s)] \cdot h \]

\[ = \sum_{s=x-h}^{s=x} (d (s) - \lambda \rho (s)) [a (s) y_x (s) + a (s+h) y_x (s)] y (s) \cdot h \]

(35)

(x > 0, x' > 0)

(the index $n$ is dropped for the time being).

From the condition of normalization $[\rho, y^2] = 1$ it follows that there exists at least one point $x'$, at which $\rho (x') y^2 (x') \leq 1$ and hence, $y^2 (x') \leq 1/c_3$. Using the Cauchy–Bunyakovskii inequality for transformation of the right-hand side of (34) and bearing (32) in mind, we shall obtain
Further from the condition \((a, y_{\lambda}^{2} \leq \lambda^{h})\) it follows that there exists a point \(x'\), at which
\[ a(x') y_{\lambda}^{2}(x') \leq \lambda^{h}, \]
i.e.
\[ (a(x') y_{\lambda}^{2}(x'))^{2} \leq c_{2} \lambda^{h}. \]

Then, by using the Cauchy–Bunyakovskii inequality for the transformation of the right-hand side of (35) and bearing in mind (25) and (32), we shall have
\[ y_{\lambda}^{2}(x) \leq \frac{c_{2}}{c_{1}^{2}} \lambda^{h} + 2 \sqrt{\frac{c_{2}}{c_{1}} \frac{c_{2}}{c_{1}} (\lambda^{h})^{1/4} + \frac{2}{c_{1}^{2}} (\lambda^{h})^{1/4} \leq M_{2}^{2} n^{2}. \] (37)

From the inequalities (36) and (37), by virtue of the arbitrary nature of \(x\), the estimates (33) follow.

Estimates (33) are rougher in comparison with estimates (7) for problem (I). However, so as not to complicate the exposition, we shall not deal with their refinement, all the more since for our purposes there is no need to do so.

The condition of normalization \(H_{N}[y] = 1\) determines the eigenfunction \(y\) with accuracy except for the sign. For unambiguous definition of the eigenfunction we must introduce a further condition to select the sign. In the case of a boundary condition when \(x = 0\) of the second or third kind, we may require for this purpose that \(y(0) > 0\), and in the case of a boundary condition of the first kind, \((y(0) = 0)\), require that \(y_{x,0} > 0\). An analogous selection of the sign may also be made for the eigenfunctions \(u(x)\) of the initial problem (I). Henceforward normalization of the eigenfunctions, side by side with the conditions \(H_{N}[y] = 1\) and \(H[u] = 1\), will also include selection of the sign by the method indicated above.

§ 2. CONVERGENCE OF SOLUTIONS OF THE DIFFERENCE PROBLEM

The convergence of the eigenvalues and eigenfunctions of the difference problem (II) to the eigenvalues and functions of the initial Sturm–Lioville problem (I) when \(N \to \infty (h \to 0)\) was proved by Courant [1] for the simplest scheme \(a = k(x-h), d = q(x), \rho = r(x)\) in the class of smooth coefficients. In this section, we shall use Courant’s method to prove convergence for problem (II) in the class of piecewise-continuous coefficients \((k, q, r \in Q^{(0)})\).

We shall prove the following

**Theorem 1.** If the scheme \(L_{h,q,r}^{(0,0,0)}\) has zero rank, then the solution \((\lambda_{n}^{h}, y_{n}(x))\) of problem (II) converges on any sequence of meshes when \(h \to 0\) to the corresponding solution \((\lambda_{n}, u_{n}(x))\) of problem (I)
\[ \lambda_{n}^{h} - \lambda_{n} = \rho(h), \quad \|y_{n} - u_{n}\|_{0} = \rho(h) \]
for any piecewise-continuous coefficients \(k, q, r \in Q^{(0)}\), satisfying condition (3).

We shall consider the case of the first eigenvalue \((n = 1)\).
Let $\varphi(x)$ be any piecewise-smooth function. It is not difficult to see that
\[
\lim_{N \to \infty} D_N [\varphi] = D [\varphi], \quad \lim_{N \to \infty} H_N [\varphi] = H [\varphi].
\]
It follows from this, that $D_N [\varphi] \leq M$, where $M$ is a constant independent of $N$ (or $h = 1/N$).

Let $y = y(x, h)$ be a mesh function which realizes the minimum of the functional $D_N [\varphi]$:
\[
\lambda^h = D_N [y]
\]
with the condition of normalization $H_N [y] = 1$.

We shall consider the sequence of mesh functions $\{y(x, h)\}$ on some sequence of meshes.

**Lemma 3.** The sequence of functions $\{y(x, h)\}$ is equally continuous and uniformly bounded.

If $x', x''$ are mesh points, then
\[
y(x'', h) - y(x', h) = \sum_{s=x'}^{x''-h} h \cdot y_x(s, h).
\]

Then, using the Cauchy-Bunyakovskii inequality and the limitation on $D_N [y]$ we obtain
\[
|y(x'', h) - y(x', h)| \leq \sqrt{\left( \frac{c_3}{2} \right)} \cdot \sqrt{|x'' - x'|} \leq M \sqrt{|x'' - x'|}. \quad (38)
\]

From the condition of normalization $H_N [y] = 1$ it follows that at least at one point $x = x'$ the inequality
\[
\rho(x') y^a(x', h) \leq 1, \quad \text{i. e.} \quad |y(x', h)| \leq 1/\sqrt{c_3}.
\]
is satisfied.

From this, and from (38) there follows the uniform bounding of the sequence $\{y(x, h)\}$:
\[
|y(x'')| \leq |y(x', h) - y(x', h)| + |y(x', h)| \leq M.
\]

According to Artzel's theorem, which is used for the sequence of mesh functions, there exists some sub-sequence $\{y(x, h_k)\}$ which converges uniformly to some function $\tilde{u}(x)$, which is continuous on the line segment $[0, 1]$: \[
\|y(x, h_k) - \tilde{u}(x)\|_0 = \rho(h_k).
\]

We shall assume that the sequence $\{h_k\}$ is such that the numerical sequence $\{\tilde{\lambda} = \lambda(h_k)\}$ converges to some limit $\tilde{\lambda}$:
\[
\lim_{h_k \to 0} \lambda(h_k) = \tilde{\lambda}.
\]
Otherwise we would have selected from it a convergent subsequence and would have considered only those numbers $k$ which correspond to this subsequence.

**Lemma 4.** For some sequence
\[
\lim_{h_k \to 0} \lambda(h_k) = \tilde{\lambda},
\]
then $\tilde{\lambda} \leq \lambda$. 
Let \( u^*(x) \) be some piecewise-smooth function, for which
\[
\lambda^* = \frac{D[u^*]}{H[u^*]} \leq \lambda + \varepsilon, \quad \varepsilon > 0,
\]
and let
\[
\lambda^* (h_k) = \frac{D_N[u^*]}{H_N[u^*]} \quad (N_k = 1/h_k).
\]
By virtue of the principle of the minimum \( \lambda(h_k) \leq \lambda^*(h_k) \) and \( \lambda^*(h_k) \to \lambda^* \) when \( h_k \to 0 \). In the limit when \( h_k \to 0 \), we obtain
\[
\tilde{\lambda} \leq \lambda^* \leq \lambda + \varepsilon.
\]
From this, by virtue of the arbitrary nature of \( \varepsilon \), it follows that \( \tilde{\lambda} \leq \lambda \).

Our immediate aim is to show that the limit function satisfies conditions (1) and (2) when \( \lambda = \tilde{\lambda} \).

As was shown in § 1, Section 6, difference problem (II) is equivalent to the equation
\[
y = [G^0, (\lambda^h, p - d) y],
\]
where \( G^0 \) is Green's function for the operator \((a_{y,x})_x\) with conditions (23). We shall now perform the limit transition to (31) when \( h_k \to 0 \) and use Lemma 1. We shall then obtain
\[
\tilde{u}(x) = \int_0^1 G_0(x, \xi) (\tilde{\lambda} r(\xi) - q(\xi)) \tilde{u}(\xi) \, d\xi,
\]
where \( G_0(x, \xi) \) is Green's function for the operator \((ku')'\) with conditions (2).

From this, by the definition of Green's function, it follows that the solution \( \tilde{u}(x) \) of the integral equation (39) satisfies the differential equation
\[
L^{(k, \varphi)} \tilde{u} + \tilde{\lambda} r \tilde{u} = 0
\]
and the boundary conditions
\[
k(0) \tilde{u}'(0) - \sigma_1 \tilde{u}(0) = 0, \quad k(1) \tilde{u}'(1) + \sigma_2 \tilde{u}(1) = 0.
\]
Since to each eigenvalue of problem (I) there corresponds only one eigenfunction \( u_1(x) \), and \( \lambda = \lambda_1 \) is the least eigenvalue, then
\[
\tilde{\lambda} = \lambda_1 \quad \text{and} \quad \tilde{u}(x) = u_1(x).
\]
It also follows from what has been said that the whole sequence \( \{y(x, h)\} \) converges uniformly to \( u(x) \) and \( \lambda_1^h = \lambda_1(h) \) converges to \( \lambda_1 \) when \( h \to 0 \):
\[
\| y_1 - u_1 \|_0 = \rho(h), \quad \lambda_1^h - \lambda_1 = \rho(h).
\]
The reasoning above related to the least eigenvalue \( \lambda_1^h \).

In the case of other eigenvalues \( \lambda_n^h \) for \( n > 1 \) all the reasoning remains valid, if we bear in mind that \( \lambda_n^h \) and \( \lambda_n \) are defined as the minimums of the functionals \( D_N[y] \) and, correspondingly, \( D[y] \), with the further conditions of the orthogonality of \( H_N[y, y_m] = 0 \) and \( H[y u_m] = 0 \) \((1 \leq m < n)\). Theorem 1 is thus proved.
3. ON THE ACCURACY OF THE DIFFERENCE METHOD

1. An equation for eigenfunction error. Let \((\lambda^h, y)\) be the solution of the difference problem (II) and \((\lambda, u)\) the corresponding solution of the initial Sturm–Liouville problem (I). We shall answer the question of the asymptotic order when \(h \to 0\) of the error \(z = y - u\) in the norm \(\|\cdot\|_0\) and also for the difference \(\Delta \lambda = \lambda^h - \lambda\). We shall first of all formulate the boundary condition for \(z\). We shall substitute \(y = z + u\) in equation (22) and bear in mind equation (1) for \(u\); we shall then obtain for \(z\) the inhomogeneous difference equation

\[(a \xi \eta) \cdot d \cdot z + \lambda^h \cdot z = -\Psi',\]

where

\[\Psi' = \psi + (\lambda^h - \lambda) \rho u,\]

\[\psi = L_h^{(k, \nu, \lambda)} u - L_h^{(k, \nu, \lambda)} u = \left[\left((au)_x - (ku')'\right) - (d - q) u + \lambda (\rho - r) u.\right]\]

Function \(\psi\) is the error of approximation of the difference scheme on the solution of equation (I).

For function \(z\) we obtain the inhomogeneous boundary conditions

\[a_1 z_{x,0} - \bar{a}_1 z_0 + h \lambda^h \rho_0 z_0 = \nu_1, \quad a_N z_{x,N} + \bar{a}_2 z_N - h \lambda^h \rho_N z_N = -\nu_2,\]

where

\[\nu_1 = h (\lambda^h - \lambda) \rho_0 u_0 + \bar{\nu}_1, \quad \nu_2 = h (\lambda^h - \lambda) \rho_N u_N + \bar{\nu}_2\]

\[\bar{\nu}_1 = (a_1 u_x (0) - \bar{a}_1 u (0) + h \lambda \rho_0 u (0)) - (k (0) u' (0) - \sigma_1 u (0)),\]

\[\bar{\nu}_2 = (a_N u_x (1) + \bar{a}_2 u (1) - h \lambda \rho_N u (1)) - (k (1) u' (1) + \sigma_2 u (1)).\]

Thus to ascertain the accuracy of the solution of the difference problem we have to estimate the solution of equation (40) with boundary conditions (43). This problem will be called problem (III).

2. Formula for \(\Delta \lambda = \lambda^h - \lambda\). Parameter \(\lambda^h\) is an eigenvalue. Therefore problem (III) is solvable only in the case when the eigenfunction \(y\) of problem (II) is orthogonal to the right-hand side of the equation and the boundary conditions, or more accurately:

\[[\psi, y] = (\psi, y) + (\lambda^h - \lambda) [\rho u, y] = 0,\]

where

\[h \psi_0 = \bar{\nu}_1, \quad h \psi_N = \bar{\nu}_2, \quad \rho_0 = 0.5 r (0), \quad \rho_N = 0.5 r (1).\]

We shall assume \(u\) and \(y\) to be normalized functions, such that

\[H[u] = 1, \quad H_N [y] = 1.\]

By virtue of Theorem 1 \(\|y - u\|_0 \to 0\) when \(h \to 0\). Therefore with a sufficiently small \(h \leq h_0\) we can assert that \([\rho u, y] \neq 0\). To the eigenvalue \(\lambda^h\) there corresponds only one eigenfunction, which is definable with accuracy up to the arbitrary factor.
We shall select the factor $C$ in such a way that the function $\bar{y} = Cy$ is orthogonal to the difference $\bar{z} = \bar{y} - u$:

$$[\rho \bar{y}, \bar{z}] = 0,$$

where $\bar{z} = \bar{y} - u$. (48)

It follows from this that

$$[\rho, yu] = [\rho, \bar{y}y] = C [\rho, y^2] = C,$$

$$[\rho, u^2] = [\rho, \bar{y}^2] + [\rho, \bar{z}^2] = C^2 + [\rho, \bar{z}u],$$

$$C^2 = [\rho, u^2] - [\rho, \bar{z}u]$$

or

$$1 - C^2 = [\rho, \bar{z}u] - (H_N[u] - H[u]).$$

(49)

By virtue of Theorem 1 it is clear that $C^2 \to 1$ when $h \to 0$. We shall assume that $C > 0$, i.e., the signs of $u$ and $y$ are coordinated (see § 1, Section 7). Formula (49) will be needed for the estimate of $|C^2 - 1|$.

We shall use condition (47) to define

$$\Delta \lambda = \lambda_h - \lambda = - [\psi, y]/[\rho u, y] = - [\psi, \bar{y}]/C^2.$$  

(50)

We shall transform the right-hand side. We introduce the function $\eta$ with the aid of the conditions

$$\eta_x = \psi, \quad \eta_0 = h\psi_0 = \bar{v}_1, \quad \eta(x) = \sum_{x'=0}^x h \psi(x').$$

Applying summation to both sides of (16)

$$[\psi, \bar{y}] = - [\eta, y_N] + y_N \eta_N,$$

and also Lemma 2, we obtain

$$|[\psi, \bar{y}]| \leq M(n) \|\psi\|_4,$$

where

$$\|\psi\|_4 = \sum_{i=1}^{N-1} h \left| \sum_{k=1}^i h\psi_k \right| + \sum_{i=0}^N h\psi_i + h \left| \psi_0 \right| + h \left| \psi_N \right|.$$  

(14)

$M(n)$ is a positive constant dependent on the number $n$ of the eigenvalue.

This proves

**Lemma 5.** If the conditions of Theorem 1 are satisfied, then

$$|\lambda_h - \lambda_n| \leq M(n) \|\psi\|_4,$$

(51)

where the function $\psi$ is defined by formula (42).

3. Integral relation for $\bar{z} = \bar{y} - u$. For estimating $\bar{z}$ we shall reduce problem (III) to the “integral” equation

$$\bar{z} = \lambda^h [G, \rho \bar{z}] + [G, \bar{y}],$$

(52)

where $G = G^h(x, \xi)$ is Green’s difference function of the operator $L_{k^h}^{[k^h, k^l]} = (a y_x)_x - dy$ with boundary conditions (23).

The eigenfunction $\bar{y}$ of problem (II) satisfies the equation

$$\bar{y} = \lambda^h [G, \rho \bar{y}].$$

(29)
We introduce the symmetrical kernel

\[ K(x, \xi) = \sqrt{\rho(x)\rho(\xi)} G(x, \xi) \quad (x \in \omega_h, \ \xi \in \omega_h) \]

(the dependence of \( G \) and \( K \) on \( h \) will not be indicated) and the new functions

\[ v(x) = \sqrt{\rho(x)} \tilde{v}(x), \quad \varphi(x) = \sqrt{\rho(x)} \tilde{\varphi}(x) \quad (\rho > 0.5c_0 > 0). \]

Then from (52) and (29') we shall obtain for the mesh functions \( v(x) \) and \( \varphi(x) \) the equations

\[ v = \lambda^h [K, v] + f, \quad f = [K, \Psi], \quad \Psi = \sqrt{\rho}, \quad (53) \]

and

\[ \varphi = \lambda^h [K, \varphi]. \quad (54) \]

It is easy to see that the condition of the orthogonality of \( \varphi \) to the right-hand side of \( f \) is automatically satisfied by virtue of condition (47) and equation (54)

\[ [\varphi, f] = [\varphi(x), [K(x, \xi), \Psi(\xi)]] = [\Psi(\xi), [K(\xi, x), \varphi(x)]] \]

\[ = \frac{1}{\lambda^h} [\Psi(\xi), \varphi(\xi)] = \frac{1}{\lambda^h} \left[ \frac{\Psi}{\sqrt{\rho}}, \frac{\sqrt{\rho}}{\sqrt{\rho}} \right] = \frac{1}{\lambda^h} [\Psi, \Psi] = 0 \]

Let \( \lambda^h = \lambda^h_n \) be the eigenvalue of number \( n \) and \( \varphi_n(x) \) its normalized eigenfunction \( ([\varphi_n, \varphi_n] = 1) \).

Condition \( [\nu, \bar{\varphi}] = 0 \) is written in the form

\[ [\varphi_n, v] = 0. \quad (55) \]

Moreover, we have

\[ [\varphi_n, f] = 0. \quad (56) \]

We shall find the resolvent \( R(x, \xi; \lambda^h_n) \), with the aid of which the solution of equation (53) is given by the formula

\[ v = f + \lambda^h_n [R, f]. \quad (57) \]

From this and from conditions (55) and (56) there follows the orthogonality of \( R \) to \( \varphi_n \):

\[ [R, \varphi_n] = 0. \quad (58) \]

This same property is possessed by the kernel

\[ K_1(x, \xi) = K(x, \xi) - \frac{\varphi_n(x)\varphi_n(\xi)}{\lambda^h_n} = \sum_{k=1}^{N} \varphi_k(x)\varphi_k(\xi) \frac{\lambda^h_k}{\lambda^h_n}. \]

The resolvent \( R \) is determined from the equation

\[ R(x, \xi; \lambda^h_n) = K_1(x, \xi) + \lambda^h_n [K_1(x, t), R(t, \xi; \lambda^h_n)] \]

and can be written in the form

\[ R = K_1 + R_1, \]

where

\[ R_1 = R_1(x, \xi; \lambda^h_n) = \sum_{k=1}^{N} \frac{\varphi_k(x)\varphi_k(\xi)}{\lambda^h_k(\lambda^h_k/\lambda^h_n - 1)}. \]
Kernel $K_n$ is bounded by virtue of the limitation of Green's function and the
eigenfunction $\varphi_n(x)$ (Lemma 2). We shall therefore have:

$$\sum_{k=1}^{N} \frac{\varphi_{k}^{2}(x)}{(\lambda_{k}^{n})^{2}} \leq [K_{1}^{2}(x, \xi), 1] \leq M,$$

$$\|R_{1}\|_{2} = [R_{1}^{2}(x, \xi; \lambda_{n}^{h}), 1] = \sum_{k=1}^{N} \frac{\varphi_{k}^{2}(x)}{(\lambda_{k}^{n})^{2}(1-\lambda_{k}^{h}/\lambda_{n}^{h})^{2}} \leq M,$$

$$\|R\|_{2} \leq \|K_{1}\|_{2} + \|R_{1}\|_{2} \leq M.$$

Resorting to formula (57), we obtain

$$\|v\|_{0} < (1+\lambda_{n}^{h}\|R\|_{2})\|f\|_{0}. \quad (59)$$

4. A priori estimates

THEOREM 2. Let $(\lambda_{n}, h_{n})$ be the eigenvalue and the normalized eigenfunction
of number $n$ of problem (II), and $(\lambda_{n}, u_{n})$ the eigenvalue and normalized eigenfunction
of number $n$ of problem (I).

If the conditions of Theorem 1 are satisfied, then at a sufficiently small $h < h_{0}$
the inequalities

$$|\lambda_{n}^{h} - \lambda_{n}| \leq M_{1}(n)\|\psi\|_{4}, \quad (51)$$

$$\|y_{n} - u_{n}\|_{0} \leq M_{2}(n)\|\psi\|_{4} + M_{1}(n)\|H_{N}[u_{n}] - H[u]\|, \quad (60)$$

are satisfied, where $M_{1}(n)$ and $M_{2}(n)$ are constants dependent on $n$ and independent
of $h$.

For proof of the theorem it is sufficient to establish inequality (60), since an
estimate for $\lambda_{n}^{h} - \lambda_{n}$ has already been obtained in Section 2 (Lemma 5).

We shall turn to inequality (59) and examine $\|f\|_{0}$, where,

$$f = [K, \varphi] + (\lambda_{n}^{h} - \lambda_{n})[K, \varphi/u] \quad \varphi = \varphi/\sqrt{r}.$$

The second addend, taken with respect to the norm $\|\phi\|_{0}$, is dominated by the
quantity $M |\lambda_{n}^{h} - \lambda_{n}|$, or, in accordance with Lemma 5, the quantity $M_{1}(n)\|\psi\|_{4}$.

We shall now transform the expression

$$[K, \varphi] = \sqrt{r}(x) \{G(x, \xi), \varphi(\xi)\}.$$

Introducing the function $\eta_{1}$, with the aid of the conditions

$$\eta_{1} = \varphi, \quad \eta_{0} = \sqrt{r}\varphi_{0} = \varphi,$$

we obtain

$$[K, \varphi] = \sqrt{r}(x) \{[G_{1}(x, \xi), \eta_{1}(\xi)] + G(\xi, 1)\psi_{1}, h - G(x, 1)\psi_{1}, h\}.$$

From this, by virtue of the boundedness of Green's function and its first differ-
ce derivatives (see § 1, Section 6) there follows

$$\|f\|_{0} \leq M(n)\|\psi\|_{4}.$$
and hence
\[ \| \bar{z} \|_0 \leq \frac{M(n)}{\sqrt{c_2}} (1 + \lambda_n^2 \| R \|_2) \| \psi \|_4 \leq M(n) \| \psi \|_4. \]  

We are interested in the difference
\[ z = y - u, \]
which is expressed by \( \bar{z} \):
\[ z = \frac{\bar{z}}{C} + \frac{1-C}{C} u, \]
\[ \| z \|_0 \leq \frac{1}{C} \| \bar{z} \|_0 + \frac{1-C^2}{C(1+C)} \| u \|_0 \leq M(\| \bar{z} \|_0 + |1-C^2|) \]
when \( h \leq h_0 \), since \( C \to 1 \) when \( h \to 0 \), and \( \| u \|_0 \) is bounded by virtue of (7).

Turning now to formula (49), we find
and hence
\[ \| z \|_0 \leq M(\bar{z}) \| \bar{z} \|_0 + M \| H_N[u_n] - H[u] \| \]
and hence
\[ \| z \|_0 \leq M(n) \| \bar{z} \|_0 + M \| H_N[u_n] - H[u] \|. \]

Then, bearing in mind estimate (61) for \( \| \bar{z} \|_0 \) we obtain inequality (60). Theorem 2 is proved.

5. The order of accuracy in the class of smooth coefficients. By virtue of Theorem 2 the order of accuracy of the solution of difference problem (II) depends on the error of approximation of the difference scheme, including the boundary conditions, and also on the error of approximation of the normalized functional \( H_N \), i.e., on the magnitude
\[ \chi = H_N[u_n] - H[u_n]. \]

The estimation of \( \psi \) with respect to the norm \( \| \psi \|_4 \) proves useful even in the class \( C^{(m)} \), since it enables us to reduce by one order the requirement of differentiability of the function \( k(x) \), and also the rank of the standard functional \( A[\bar{k}(x)] \). In fact, as \emph{a priori} estimates (51) and (60) show, for the difference scheme to have an \( m \)th (\( m = 1,2 \)) order of accuracy, it is sufficient for the condition \( \| \bar{z} \|_0 = O(h^m) \) to be satisfied. At the same time the scheme may also fail to have an \( m \)th order of approximation, i.e., the condition \( \psi = O(h^m) \) will not be satisfied.

It will be shown below that \( \| \bar{z} \|_0 = O(h^m) \) in the class \( k, q, r \in C^{(m-1,1)} \), if the scheme has rank \( m \) and satisfies conditions (20) for an \( m \)th order of approximation.

In § 1, Section 3, we agreed to consider only schemes of the standard type. We would also recall that \( D[f] \) and \( R[f] \) are linear functionals.

THEOREM 3. If the difference scheme \( L_h^{(k,q,r)} \) has \( 2 \)nd rank and satisfies the conditions (21) necessary for the second order of approximation, then the solution of problem (II) for \( k, q, r \in C^{(1,1)} \) has a second order of accuracy:
\[ |\lambda_n^2 - \lambda_n| \leq M(n) h^2, \quad \| y_n - u_n \|_0 \leq M(n) h^2. \]
For proof of the theorem it is sufficient to estimate \( \| \psi \|_4 \) and \( \chi \) and use Theorem 2, the conditions of which are satisfied.

We shall consider the error of approximation
\[
\psi = \varphi - (d-q)u + \lambda (\varphi - r)u
\]
(the index \( n \) is dropped), where
\[
\varphi = (au_x)_x - (ku')' = \mathcal{L}u - \sum_{k=0}^{h} u.
\]

If the conditions of the theorem are satisfied, there exist derivatives \( u'' \) and \((ku')''\), satisfying the Lipschitz condition.

We shall therefore have
\[
(ku')' = (ku')_x + o(h^2),
\]
where the bar above denotes that the expression is taken at the point \( \bar{x} = x - 0.5h \).

Let us first consider
\[
\sum_{x'=x-h}^{x'=x-h} h\varphi(x') = \sum_{x'=x-h}^{x'=x-h} \left( a(x') u_x(x') - k(x') u'(x') \right) h + O(h^2)
\]
\[
= a(x) u_x(x) - k(\bar{x}) u'(\bar{x}) - \left( a(x') u_x(x') - k(\bar{x}) u'(\bar{x}) \right) \bigg|_{x'-h} + O(h^2).
\]

By virtue of conditions (21) \( a = k + O(h^2) \). Then, bearing in mind that \( u_x = \bar{u}' + O(h^2) \) (since \( u'' \) satisfies the Lipschitz condition), we obtain \( au_x \bar{u}' = O(h^2) \), and hence
\[
\sum_{x'=x-h}^{x'=x-h} h\varphi(x') = O(h^2), \quad \| \varphi \|_3 = O(h^2).
\]

The difference boundary conditions (23), as we saw in § 1, Section 4, have a second order of approximation in the class of solutions of equation (1), i.e., \( \bar{v}_1 = O(h^2) \) and \( \bar{v}_2 = O(h^2) \). It follows from this that
\[
\| \varphi \|_4 = O(h^2), \quad \text{more accurately,} \quad \| \varphi \|_4 = M \cdot h^2.
\]

Then, bearing in mind that by virtue of conditions (20) \( d-q = O(h^2) \), \( \varphi - r = O(h^2) \), we shall have
\[
\| \psi \|_4 \leq M(n) h^2.
\]

Noting that \( H[u] \) is a quadrature formula for \( H[u] \) of the second order of accuracy in the case \( r \in C^{(1,1)} \) and \( u \in C^{(1,1)} \) we find
\[
\chi = O(h^2).
\]

Theorem 3 is proved.

6. The order of accuracy in the class of discontinuous coefficients. We shall now assume that \( k, q, r \in Q^{(1,1)}[0,1] \); then \( u' \in Q^{(1,1)} \), \( (ku')' \in Q^{(1,1)} \). We shall denote \( \xi_j = x_n + \delta_j h(x_n = h n, 0 \leq \delta_j \leq 1, 0 < \xi_j < 1) \) all the points of discontinuity of the functions \( k(x), q(x) \) and \( r(x) \). The number \( j_0 \) of such discontinuities is finite: \( j = 1, 2, \ldots, j_0 \).
In computing \( \psi \) in this case we shall refer to § 3 of the paper [4]. Let \( L_h^{k, q, \lambda} \) be a scheme of 2nd rank, satisfying conditions (21). We shall represent \( \psi \) in the form of the sum
\[
\psi = \bar{\psi} + \overline{\psi},
\]
where
\[
\bar{\psi} = \bar{\psi}_j = \psi_{n_j} \delta_{i,n_j} + \psi_{n_j+1} \delta_{i,n_j+1},
\]
\( j = 1, 2, \ldots, J_0 \),
\[
\delta_{i,k} = \begin{cases} 0, & i \neq k \\ 1, & i = k. \end{cases}
\]
By the same reasoning as in the proof of Theorem 3, we obtain
\[
\| \bar{\psi} \|_4 = O(h^2), \quad \text{more accurately} \quad \| \bar{\psi} \|_4 \leq M(n) h^2.
\] (62)
It must be borne in mind here that \( h_{\bar{\psi}_0} = \bar{\psi}_1, h_{\bar{\psi}_N} = \bar{\psi}_2 \).

We now turn to the computation of \( \| \bar{\psi} \|_4 \). We shall note first that for every scheme of the above-mentioned family, the conditions
\[
h_{\psi_{n_j}} = O(1), \quad h_{\psi_{n_j+1}} = O(1), \quad h (\psi_{n_j} + \psi_{n_j+1}) = O(h),
\]
\( j = 1, 2, \ldots, J_0 \).

For the difference scheme
\[
L_h^{k, q, \lambda} y = (ay)_{x} - d \cdot y + \lambda \cdot \rho y
\] (63)
with coefficients
\[
a = \left[ \int_{-\frac{1}{2}}^{0} \frac{ds}{k(x+sh)} \right]^{-1}, \quad d = \int_{-\frac{1}{2}}^{0.5} q(x+sh) \, ds, \quad \rho = \int_{-\frac{1}{2}}^{0.5} r(x+sh) \, ds
\] (64)
the conditions [4]
\[
h_{\psi_{n_j}} = O(h), \quad h_{\psi_{n_j+1}} = O(h), \quad h (\psi_{n_j} + \psi_{n_j+1}) = O(h^2), \quad j = 1, 2, \ldots, J_0.
\]
are satisfied.

It is easy to see that
\[
\| \bar{\psi} \|_3 \leq \sum_{j=1}^{J_0} (h^2 | \psi_{n_j} | + h | \psi_{n_j} + \psi_{n_j+1} |),
\]
\[
\| \bar{\psi} \|_4 \leq \sum_{j=1}^{J_0} (h^2 | \psi_{n_j} | + 2h | \psi_{n_j} + \psi_{n_j+1} |).
\]

**Lemma 6.** If
\[
\rho = \rho_1 = R_1[r(x+sh)] = \int_{-\frac{1}{2}}^{0.5} r(x+sh) \, ds,
\]
then for \( r(x) \in Q^{(0,1)} \) and \( u \in C^{(1)}, u' \in Q^{(1,1)} \) the estimate
\[
\chi = H_N[u] - H[u] = O(h^2).
\]
To simplify the notation, without disrupting the generality, we can consider that there is only one point \( \xi = x_n + \theta h \) of discontinuity of the function \( r(x) \). We shall represent \( \chi \) in the form of a sum:

\[
\chi = \sum_{i=0}^{N} \Delta_i \cdot h,
\]

where

\[
\Delta_i = \int_{-0.5}^{0.5} r(x_i + sh) \left[ u^2(x_i) - u^2(x_i + sh) \right] ds, \quad 0 < i < N,
\]

\[
\Delta_0 = 0.5hr(0) u^2(0) - \int_0^1 r(x) u^2(x) \, dx,
\]

\[
\Delta_N = 0.5hr(1) u^2(1) - \int_{1-0.5h}^{1} r(x) u^2(x) \, dx.
\]

It can immediately be seen that \( \Delta_0 = O(h^2) \), \( \Delta_N = O(h^2) \). If \( i \neq n, i \neq n+1 \), then

\[
\Delta_i = -\int_{-0.5}^{0.5} \left( r(x_i) + shr(x_i) + h\phi(h) \right) \left( u^0, sh + O(h^2) \right) ds = O(h^2).
\]

We shall now assume that \( 0 < \theta < 0.5 \). Then

\[
\Delta_n = \int_{-0.5}^{0} \left( r_0 + O(h) \right) \cdot O(h) \, ds + \int_{0}^{0.5} \left( r_0 + O(h) \right) \cdot O(h) \, ds = O(h),
\]

\[
r_0 = r(\xi - 0), \quad r_0 = r(\xi + 0), \quad \Delta_{n+1} = O(h^2).
\]

If, however, \( 0.5 \leq \theta \leq 1 \), then

\[
\Delta_n = O(h^2), \quad \Delta_{n+1} = O(h).
\]

In both cases \( \Delta_n + \Delta_{n+1} = O(h) \). It follows from this that

\[
\chi = h(\Delta_n + \Delta_{n+1}) + O(h^2) = O(h^2).
\]

**Corollary.** If \( R[\hat{r}(s)] \) is an arbitrary functional, then

\[
\chi = [\rho - \rho_1, u^0] + O(h^2) \quad \text{for} \quad r \in Q^{(0.1)}.
\]

**Note.** Using the representation of linear functionals in the class of discontinuous coefficients (see § 1, Section 11, reference [4]) it can be shown that there exists only one linear canonical normalized functional, for which \( \chi = O(h^2) \) in the class \( Q^{(m)} \) for any \( m \geq 1 \).

We have thus proved the following theorems

**Theorem 4.** For any difference scheme of second rank, satisfying conditions (21) in the class \( Q^{(1.1)} (k, q, r \in Q^{(1.1)}) \) the relations

\[
|\lambda_n - \lambda_0| \leq M(n)h, \quad \|y_n - u_n\|_0 \leq M(n) \cdot h,
\]
are satisfied, where \( \lambda_n, u_n(x) \) are the \( n \)-th eigenvalue and the \( n \)-th normalized eigenfunction of problem (I) and \( \lambda_n^k, y_n \) the \( n \)-th eigenvalue and \( n \)-th eigenfunction of the Sturm–Liouville difference problem (II).

**Theorem 5.** The difference scheme (63)-(64) ensures in the class of coefficients \( k, q, r \in Q^{(1,1)} \) the second order of accuracy:

\[
|\lambda_n^k - \lambda_n| \leq M(n) h^2, \quad \|y_n - u_n\|_0 \leq M(n) h^2.
\]

At a later date we shall consider homogeneous difference schemes, giving any order of accuracy in the class of piecewise-continuous coefficients of equation (1).

Separate consideration will also be given to the question of accuracy on non-uniform meshes.

*Translated by G. K. Elliott*

**References**

5. COURANT R., and HILBERT D., Methods of Mathematical Physics, Vol. 1, 1951.