THE STURM-LIOUVILLE DIFFERENCE PROBLEM*

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THE solution of the Sturm-Liouville problem for the equation

$$L^{(k,q)}u+\lambda r(x)u=0, \qquad 0<x<1, \qquad L^{(k,q)}u=\frac{\mathrm{d}}{\mathrm{d}x}\left[k(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right]-q(x)u(x) \tag{1}$$

by the method of finite differences has been the subject of a great many studies. The question of convergence and accuracy in the class of smooth coefficients for difference schemes of a particular form were considered in [1]–[3].

In this paper we employ the homogeneous difference schemes, studied in [4], for solving the Sturm–Liouville problem in the class of discontinuous coefficients $Q^{(m)}$. The formulation of the problem and the characteristics of the initial family of difference schemes are given in § 1. In § 2 the convergence of the difference method is proved. In § 3, with the aid of *a priori* estimates we establish the order of accuracy in $Q^{(m,1)}$ of the solution of the difference problem when $h \to 0$. It is shown that the difference scheme

$$L_h^{(k,q,\lambda r)}y=(ay_{\bar{x}})_x-dy+\lambda\rho y,$$

where

$$a=\left[\int_{-1}^{0}\frac{\mathrm{d}s}{k(x+sh)}\right]^{-1}, \qquad d=\int_{-0\cdot5}^{0\cdot5}q(x+sh)\,\mathrm{d}s, \qquad \rho=\int_{-0\cdot5}^{0\cdot5}r(x+sh)\,\mathrm{d}s,$$

ensures the second order of accuracy in the class of discontinuous coefficients.

§ 1. DIFFERENCE BOUNDARY-VALUE PROBLEM

1. *Sturm–Liouville problem.* We shall consider the homogeneous differential equation (1) with homogeneous boundary conditions

$$k(0)\,u'(0)+\sigma_1u(0)=0, \qquad k(1)\,u'(1)-\sigma_2u(1)=0, \tag{2}$$

where σ_1 and σ_2 are some constants.

The Sturm–Liouville problem, or the eigenvalues problem, consists in finding those values of the parameter λ (eigenvalues) at which there exist non-trivial solutions (eigenfunctions) of the problem (1)–(2) and also in finding the eigenfunctions.

Henceforward we shall everywhere assume that the coefficients entering into equation (1) and the condition (2) satisfy the conditions

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$$0 < c_1 \leqslant k(x) \leqslant c_2, \quad 0 < c_3 \leqslant r(x) \leqslant c_4, \quad 0 \leqslant q(x) \leqslant c_5,$$
$$\sigma_1 \geqslant 0, \quad \sigma_2 \geqslant 0, \quad \sigma_1 + \sigma_2 > 0, \tag{3}$$

where $c_j\,(j = 1, \ldots, 5)$ are some constants.

If $k(x)$ has a discontinuity of the first kind at the point $x = \xi (0 < \xi < 1)$, then at this point the conditions of conjugation (continuity of $u(x)$ and $k(x)\,u'(x)$) must be satisfied:

$$[u] = 0, \quad [ku'] = 0 \quad \text{when} \quad x = \xi, \tag{4}$$

where

$$[f] = f_r - f_l, \quad f_l = f(\xi - 0), \quad f_r = f(\xi + 0).$$

The problem defined by condition (1)–(4) will henceforward be termed problem (I).

We shall use the following notations: $Q^m\,[l_1, l_2]$ is a class of functions which are piecewise-continuous on the line segment $[l_1, l_2]$ together with their derivatives to the mth order inclusive; $Q^{(m,\gamma)}\,[l_1, l_2]\,(0 \leqslant \gamma \leqslant 1)$ is a class of functions from $Q^{(m)}\,[l_1, l_2]$, the mth derivatives of which satisfy on the intervals of their continuity the Hölder condition of the order γ; $C^{(m)}\,[l_1, l_2]$ is, as usual, a class of functions having an mth continuous derivative.

As we know, problem (I) is equivalent to the variation problem:

(1) In the class of piecewise-smooth comparison functions $\varphi(x)$, satisfying the conditions

$$H[\varphi] = \int_0^1 \varphi^2(x)\, r(x)\, dx = 1,$$

$$k(0)\,\varphi'(0) - \sigma_1 \varphi(0) = 0, \quad k(1)\,\varphi'(1) + \sigma_2 \varphi(1) = 0,$$

find the minimum of the functional

$$D[\varphi] = \int_0^1 k(x)\,(\varphi')^2 dx + \int_0^1 q(x)\,\varphi^2(x)\, dx + \sigma_1 \varphi^2(0) + \sigma_2 \varphi_2(1). \tag{5}$$

This minimum determines the first eigenvalue

$$\lambda_1 = \min D[\varphi] = D[u_1]$$

(2) The remaining eigenvalues $\lambda_n (n > 1)$ are found as the minimum of functional (5) in the class of piecewise-smooth comparison functions $\varphi(x)$ satisfying the additional conditions

$$H[\varphi] = 1, \quad H[\varphi, u_m] = \int_0^1 \varphi(x)\, u_m(x)\, r(x)\, dx = 0 \quad \text{when} \quad m < n,$$

$$k(0)\,\varphi'(0) - \sigma_1 \varphi(0) = 0, \quad k(1)\,\varphi'(1) + \sigma_2 \varphi(1) = 0,$$

where $u_m(x)$ is the eigenfunction of the number m. This minimum determines the nth eigenvalue

$$\lambda_n = \min D[\varphi] = D[u_n]$$

and is reached at the nth eigenfunction $u_n(x)$.

The Sturm–Liouville problem (I) for piecewise-continuous coefficients k, q, $r \in Q^{(0)}$ has the denumerable set of eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots$, to which there correspond the eigenfunctions $u_1(x), u_2(x), \ldots, u_n(x), \ldots$. (see [5]).

We shall indicate certain known [5] properties of eigenfunctions and eigenvalues.

1. To each eigenvalue there corresponds only one eigenfunction.

Let us assume that to λ_n there correspond two eigenfunctions $\bar{u}_n(x)$ and $\bar{\bar{u}}_n(x)$. Then their linear combination $\tilde{u}_n(x) = \bar{\bar{u}}_n(0)\bar{u}_n(x) - \bar{u}_n(0)\bar{\bar{u}}_n(x)$ satisfies the condition $\tilde{u}_n(0) = 0$ and is also the eigenfunction corresponding to λ_n. From the condition $k(0)\tilde{u}'(0) - \sigma_1\tilde{u}(0) = 0$ for $\sigma_1 \neq \infty$ we obtain $\tilde{u}'(0) = 0$. It follows from this that $\tilde{u}_n(x) \equiv 0$. In the case of the condition of the first kind ($\sigma_1 = \infty$) one can select $\tilde{u}_n(x) = \bar{\bar{u}}_n'(0)\,\bar{u}_n(x) - \bar{u}_n'(0)\,\bar{\bar{u}}_n(x)$. We can therefore write $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$.

2. Eigenfunctions $\{u_n(x)\}$ form an orthogonal system normalized with weight $r(x)$:

$$H[u_n,\ u_m] = 0 \quad \text{when} \quad m \neq n, \quad H[u_n] = 1.$$

3. All eigenvalues are positive: $\lambda_n > 0$, $n = 1, 2, \ldots$.

4. Eigenvalues $\lambda_n \to \infty$ when $n \to \infty$, or, more accurately,

$$c_6 n^2 \leqslant \lambda_n \leqslant c_7 n^2, \tag{6}$$

where c_6 and c_7 are positive constants independent of the number n and dependent solely on $c_j (j = 1, \ldots, 5)$.

5. Eigenfunctions and their first derivatives are bounded — more accurately,

$$|u_n(x)| < c_8, \quad |u_n'(x)| \leqslant c_9' \sqrt{\lambda_n} \leqslant c_9 n, \tag{7}$$

where c_8 and c_9 are positive constants dependent solely on $c_j (j = 1, \ldots, 5)$.

For the case $k, q, r \in C^{(2)}$ proof of (7) is given in [5]. This proof may also be transferred, with certain changes, to the case $k, q,\ r \in Q^{(0)}$.

Let us introduce the new variable $t = \int\limits_0^x r(x)\,\mathrm{d}x$. Then equation (1) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\bar{k}(t)\frac{\mathrm{d}\bar{u}}{\mathrm{d}t}\right] - \bar{q}(t)\,\bar{u} + \lambda\bar{u} = 0, \quad 0 < t < l, \tag{1'}$$

where

$$\bar{k}(t) = k(x)\,r(x), \quad \bar{q}(t) = q(x)/r(x), \quad \bar{u}(t) = u(x), \quad l = \int\limits_0^1 r(x)\,\mathrm{d}x.$$

We multiply equation (1′) by $\bar{u}'(t)$ and integrate from 0 to t:

$$\bar{k}(0)\,[\bar{u}'(0)]^2 + \lambda\bar{u}^2(0) = \bar{k}(t)\,[\bar{u}'(t)]^2 + \lambda\bar{u}^2(t) - 2\int\limits_0^t \bar{q}\bar{u}\bar{u}'\,\mathrm{d}t_1. \tag{8}$$

We integrate once more with respect to t from 0 to l, and bear in mind that

$$\int_0^l \bar{u}^2(t)\,\mathrm{d}t = 1, \quad \int_0^l \bar{k}(t)[\bar{u}'(t)]^2\,\mathrm{d}t \leqslant \lambda,$$

$$2\left|\int_0^l \mathrm{d}t\left(\int_0^l \bar{q}\bar{u}\bar{u}'\,\mathrm{d}t_1\right)\right| \leqslant 2l \cdot c_5\left(\int_0^l\left[\bar{u}^2\mathrm{d}t\int_0^l[\bar{u}'(t)]^2\,\mathrm{d}t\right)^{\frac{1}{2}} \leqslant \frac{2lc_5}{c_1c_3}\sqrt{\lambda} \leqslant c_{10}\sqrt{\lambda}. \tag{9}$$

As a result we will have

$$\bar{k}(0)[\bar{u}'(0)]^2+\lambda\bar{u}^2(0) \leqslant \frac{2\lambda}{l}+c_{10}\sqrt{\lambda}. \tag{10}$$

Returning now to (8) and bearing in mind inequalities (9) and (10) we obtain

$$\bar{k}(t)[\bar{u}'(t)]^2+\lambda\bar{u}^2(t) \leqslant \frac{2\lambda}{l}+c_{10}\sqrt{\lambda}.$$

From this, and from (6) there follow estimates (7).

Problem (I) is equivalent to an integral equation with synthetizable kernel:

$$u(x) = \lambda\int_0^1 G(x,\xi)\,r(\xi)\,u(\xi)\,\mathrm{d}\xi, \tag{11}$$

where $G(x,\xi)$ is Green's function for the operator $L^{(k,q)}$ with boundary conditions (2), and also to the integral equation

$$u(x) = \int_0^1 G_0(x,\xi)\,\big(\lambda r(\xi)-q(\xi)\big)\,u(\xi)\,\mathrm{d}\xi, \tag{12}$$

where $G_0(x,\xi)$ is Green's function for the operator $L^{(k)}u = (ku')'$ with boundary conditions (2).

2. *Notations.* Let us consider on the line segment [0,1] the difference mesh

$$\omega_h = \{x_0 = 0, \ldots, x_i = i \cdot h, \ldots, x_N = N \cdot h = 1\}.$$

The mesh function y_i, defined by ω_h, will henceforward be designated by y or $y(x)$, when this will not give rise to misunderstanding. We shall also write

$$\overset{(+1)}{y} = y_{i+1}, \quad \overset{(-1)}{y} = y_{i-1}, \quad y_{\bar{x}} = (y-\overset{(-1)}{y})/h, \quad y_x = (\overset{(+1)}{y}-y)/h.$$

Let v be some mesh function defined on ω_h. We shall introduce the notations

$$(y,v) = \sum_{i=1}^{N-1} y_iv_ih, \quad [y,v) = \sum_{i=0}^{N-1} y_iv_ih, \quad (y,v] = \sum_{i=1}^{N} y_iv_ih, \quad [y,v] = \sum_{i=0}^{N} y_iv_ih.$$

We shall use the following norms:

$$\|v\|_0 = \max_{\omega_h}|v_i|, \quad \|v\|_\sigma = [|v|^\sigma, 1]^{1/\sigma} \qquad (\sigma = 1, 2),$$

$$\|v\|_3 = \sum_{i=1}^{N-1} h\left|\sum_{k=1}^{i} hv_k\right|, \tag{13}$$

$$\|v\|_4 = \|v\|_3+|(v,1)|+|v_0| \cdot h+|v_N|\,h. \tag{14}$$

It may be found that the function v is defined not on the whole mesh, but on part of it. For example, the function $y_{\bar{x}}$ is defined at the points x_i for $i = 1, 2, \ldots, N$, and the function y_x—for $i = 0, 1, 2, \ldots, N-1$. In this case

$$\|y_{\bar{x}}\|_0 = \max_{0<i\leqslant N} |y_{\bar{x},\,i}|, \quad \|y_{\bar{x}}\|_\sigma = (|y_{\bar{x}}|^\sigma, 1]^{1/\sigma}, \qquad \sigma = 1, 2.$$
$$\|y_x\|_0 = \max_{0\leqslant i<N} |y_{x,\,i}|, \quad \|y_x\|_\sigma = [|y_x|^\sigma, 1)^{1/\sigma}.$$

We shall also write

$$[y, \psi] = (y, \psi) + y_0\bar{\nu}_1 + y_N\bar{\nu}_2, \tag{15}$$

where ψ is a function defined at the points $x_i, i = 1, 2, \ldots, N-1$, if we formally put $h\psi_0 = \bar{\nu}_1, h\psi_N = \bar{\nu}_2$.

The difference operator $(a_{i+1}(y_{i+1}-y_i)-a_i(y_i-y_{i-1}))/h^2$ with the aid of the notations introduced, will be written in the form $(ay_{\bar{x}})_x$.

We shall need henceforward:

(1) the formula of summation by parts:

$$[y, v_x) = -(v, y_{\bar{x}}] + (yv)_N - (yv)_0; \tag{16}$$

(2) Green's difference formulae

$$\big((ay_{\bar{x}})_x, v\big) = -(a, y_{\bar{x}}v_{\bar{x}}] + (ay_{\bar{x}}v)_N - (a^{(+1)}y_x\, v)_0, \tag{17}$$

$$\big((ay_{\bar{x}})_x, v\big) = \big((av_{\bar{x}})_x, y\big) + a_N\,(y_{\bar{x}}v - yv_{\bar{x}})_N - a_1\,(y_x\, v - yv_x)_0. \tag{18}$$

We shall not deal with the derivation of these formulae, since it is elementary. We shall designate by $\rho(h)$ any expression, uniformly converging to zero when $h \to 0$.

All constants independent of h will be designated by the letter M, without, as a rule, indicating their structure or their relationship to other constants.

3. *Difference schemes.* In deriving a difference scheme for the solution of problem (I) we shall use the results of [4]. We shall introduce the notations

$$L^{(k,\,q,\,\lambda r)}u = (ku')' - qu + \lambda ru, \qquad L_h^{(k,\,q,\,\lambda r)}y = (ay_{\bar{x}})_x - dy + \lambda\rho y.$$

We shall assume that $L_h^{(k,q,\lambda r)}$ is a homogeneous conservative scheme of the standard type [4]. Its coefficients a, d and ρ are expressed with the aid of the *standard* functionals

$$A^h[f(s)], \quad f \in Q^{(0)}[-1; 0]; \quad D^h[f(s)], \quad f \in Q^{(0)}[-0{\cdot}5; 0{\cdot}5];$$
$$R^h[f(s)], \quad f \in Q^{(0)}[-0{\cdot}5; 0{\cdot}5],$$

defined in the class $Q^{(0)} (f \in Q^{(0)})$, in terms of the coefficients k, q, r of the differential equation (1). Each coefficient of the scheme depends on only one coefficient of the differential equation (standard–type scheme):

$$a = A^h[k(x+sh)], \quad -1 \leqslant s \leqslant 0; \quad d = D^h[q(x+sh)], \quad -0{\cdot}5 \leqslant s \leqslant 0{\cdot}5;$$
$$\rho = R^h[r(x+\rho h)], \quad -0{\cdot}5 \leqslant s \leqslant 0{\cdot}5.$$

(The dependence of the coefficients a, d and ρ on h is not explicitly indicated).

As the initial family of schemes we shall consider conservative schemes, since, as shown in [4], non-conservative schemes

$$\bar{L}_h^{(k,q)} y = (by_x - ay_{\bar{x}})/h - dy$$

do not give convergence in $Q^{(m)}$. In the class of smooth coefficients the scheme $\bar{L}_h^{(k,q)}$ can be transformed into the conservative scheme

$$L_h^{(k,q)} y_i = \mu_i \bar{L}_h^{(k,q)} y_i = (\bar{a} y_{\bar{x}})_{x,i} - \bar{d}_i y_i,$$

where

$$\mu_i = \prod_{k=1}^{i-1} (b_k/a_{k+1}), \quad \bar{a}_i = a_i \mu_i, \quad \bar{d}_i = d_i \mu_i$$

(see [4]).

If the scheme $\bar{L}_h^{(k,q)}$ has an mth ($m = 1, 2$) order of approximation and $k(x) \in C^{(m+1)}$, then $\mu_i = 1 + O(h^m)$. From this it follows that the results obtained henceforth for $k(x) \in C^{(m+1)}$, $q, r \in C^{(m)}$ will also be transferable to non-conservative schemes.

We note that the authors of [1]–[3] only studied discrete schemes of a particular form (e.g., $a = k(x-h)$, $a = k(x - 0{\cdot}5h)$, $d = q$, $\rho = r(x)$), whilst in addition discrete schemes of the fourth order in $C^{(m)}$ were studied in [3].

The difference scheme $L_h^{(k,q,\lambda r)}$ is characterized by an approximation error in the class of differentiable coefficients

$$\psi = L_h^{(k,q,\lambda r)} u - L^{(k,q,\lambda r)} u,$$

where u is any function differentiable a sufficient number of times.

In studying convergence and accuracy, we shall be dealing with the approximation error on the solution $u(x)$ of equation (1).

In [5] the concept of the rank of *standard* functionals was introduced.

If all the *standard* functionals have a rank $m (m = 0, 1, 2)$, we say that $L_h^{(k,q,\lambda r)}$ is a scheme of mth rank. The requirement of definiteness of the order of approximation leads to certain conditions (which will not be written out here) to be satisfied by the standard functionals and their differentials with respect to h and with respect to the functional argument. We shall indicate two properties of standard functionals which are fulfilled for schemes of all ranks, beginning with a scheme of zero rank:

(a) $A^h[1] = D^h[1] = R^h[1] = 1$ (normalization condition)

(b) the functionals $A^h[f]$, $D^h[1]$ and $R^h[f]$ are non-decreasing: $A^h[f_2] \geqslant A^h[f_1]$ if $f_2 \geqslant f_1$, and so on.

Moreover, we assume that $D^h[f]$ and $R^h[f]$ are linear functionals.

From this, and from (3) in particular it follows that the coefficients a, d and ρ satisfy the conditions

$$0 < c_1 \leqslant a \leqslant c_2, \quad 0 < c_3 \leqslant \rho \leqslant c_4, \quad 0 \leqslant d \leqslant c_5, \tag{19}$$

where $c_j (j = 1, \ldots, 5)$ are constants entering into condition (3)

As the initial class of difference schemes $L_h^{(k,q,\lambda r)}$ we shall consider in § 2 schemes of zero rank.

In § 3 we shall also examine the family of difference schemes of the second rank satisfying the conditions of the second order of approximation (see [4]). This family belongs to the initial class, and we shall call it the initial family of schemes of the second order.

If the standard functionals are canonical, i.e., do not depend on the parameter h, the difference scheme is called canonical.

Two schemes are termed equivalent with respect to the order of approximation (accuracy) if they have an identical order of approximation (accuracy). It is not difficult to see that any scheme of mth rank is equivalent to its canonical part [4], i.e., to a scheme with canonical *standard* functionals $A^{(0)}$, $D^{(0)}$ and $R^{(0)}$ which are the main terms in the expansion of A^h, D^h and R^h with respect to h.

It follows from this that the entire exposition may be made for cannonical schemes.

We shall give more detailed characteristics of the properties of the canonical standard functionals $A[f]$, $D[f]$ and $R[f]$.

The linear canonical functionals $D[f]$ and $R[f]$, satisfying conditions (a) and (b) have any rank, as high as one wishes.

The canonical functional $A[f]$, which is generally speaking mon-linear, has mth rank if, apart from conditions (a) and (b), it satisfies two further requirements:

(c) $A[f]$ is an homogeneous functional of the first degree, i.e.,

$$A[cf] = cA[f],$$

where c is any positive number:

(d) $A[f]$ has an mth differential, so that in particular we may write

$$\begin{aligned}
&A[1+\delta\cdot f] = 1+\rho(\delta) && (\text{if } |f| \leqslant M) \text{ when } m=0,\\
&A[1+\delta\cdot f] = 1+\delta A_1[f]+\delta\rho(\delta) && \text{when } m=1,\\
&A[1+\delta\cdot f] = 1+\delta A_1[f]+\delta^2 A_2[f]+\delta^2\rho(\delta) && \text{when } m=2,
\end{aligned}$$

where $\rho(\delta)\to 0$ when $\delta\to 0$, $A_1[f]$ is a linear functional and $A_2[f]$ a quadratic functional.

The necessary conditions for the mth order of approximation of the scheme $L_h^{(k,q,\lambda r)}$ have the form $(k, q, r \in C^{(m)}[0,1])$

$$a = k(\bar{x})+O(h^m), \quad d = q(x)+O(h^m), \quad \rho = r(x)+O(h^m), \quad \bar{x} = x-0\cdot 5h, \quad m = 1, 2. \tag{20}$$

Any canonical scheme of the first rank satisfies these conditions when $m = 1$.

A canonical scheme of second rank satisfies the conditions (20) when $m = 2$, if

$$A_1[s] = -0\cdot 5, \quad D[s] = 0, \quad R[s] = 0\cdot 5. \tag{21}$$

It is easy to show that for any symmetrical scheme of the second rank conditions (21) are satisfied.

4. *Difference boundary-value problem.* Before formulating the difference analogue of problem (1) we have to formulate difference boundary conditions corresponding to conditions (2).

We shall consider the boundary operator

$$l^{(1)}u = k(0)\,u'(0) - \sigma_1 u(0)$$

and its difference analogue

$$\bar{l}_h^{(1)}y = a_1 y_{x,0} - \sigma_1 y_0.$$

It is easy to show that the operator $\bar{l}_h^{(1)}$ has only a first order of approximation. In fact, let $L_h^{(k)}$ be a scheme of second rank, satisfying the conditions (21) for the second order of approximation, so that $a = k(x-0{\cdot}5h)+O(h^2)$ for $k \in C^{(2)}$.

We shall then have

$$a_1 = k(0)+0{\cdot}5hk'(0)+O(h^2), \quad u_{x,0} = u'(0)+0{\cdot}5hu''(0)+O(h^2),$$
$$\bar{l}_h^{(1)}u - l^{(1)}u = 0{\cdot}5h\,(ku')'\,|_{x=0}+O(h^2) = 0{\cdot}5h\big(q(0)-\lambda r(0)\big)\,u(0)+O(h^2),$$

where $u(x)$ is a solution of equation (1).

It follows from this that the difference operator

$$l_h^{(1)}y = a_1 y_{x,0} - \bar{\sigma}_1 y + 0{\cdot}5h\lambda r(0)\,y_0, \qquad \text{where } \sigma_1 = \sigma_1 + 0{\cdot}5hq(0),$$

has a second order of approximation in the class of solutions of equation (1).

The same property is possessed by the operator

$$l_h^{(2}y = a_N y_{\bar{x},N} + \bar{\sigma}_2 y_N - 0{\cdot}5h\lambda r(1)\,y_N, \qquad \text{where } \bar{\sigma}_2 = \sigma_2 + 0{\cdot}5hq \quad (1),$$

which corresponds to the operator

$$l^{(2)}u = k(1)\,u'(1)+\sigma_2 u(1).$$

The Sturm–Liouville difference problem will be formulated as follows.

We are required to find those values of the parameter λ^h (eigenvalues) to which there correspond non-trivial solutions (eigenfunctions) of the homogeneous equation

$$(ay_{\bar{x}})_x - dy + \lambda^h \rho y = 0 \quad \text{at the points} \quad x = x_i, \qquad i = 1, 2, \ldots, N-1, \tag{22}$$

with homogeneous boundary conditions

$$a_1 y_{x,0} - \bar{\sigma}_1 y_0 + h\lambda^h \rho_0 y_0 = 0, \qquad a_N y_{\bar{x},N} + \bar{\sigma}_2 y_N - h\lambda^h \rho_N y_N = 0,$$

where

$$\rho_0 = 0{\cdot}5r(0), \quad \rho_N = 0{\cdot}5r(1), \quad \bar{\sigma}_1 = \sigma_1 + 0{\cdot}5hq(0), \quad \bar{\sigma}_2 = \sigma_2 + 0{\cdot}5hq(1), \tag{23}$$

and also to find these non-trivial solutions (eigenfunctions).

The coefficients of the problem satisfy the conditions

$$0 < c_1 \leqslant a \leqslant c_2, \quad 0 < c_3 \leqslant \rho \leqslant c_4, \quad 0 \leqslant d \leqslant c_5, \quad \sigma_1 \geqslant 0, \quad \sigma_2 \geqslant 0,$$
$$\sigma_1 + \sigma_2 > 0. \tag{24}$$

In the neighbourhood of discontinuity of the coefficient k, the conditions of continuity, like conditions (4), are absent, since we are considering homogeneous schemes, which do not envisage the explicit isolation of the points of discontinuity of the coefficients of equation (1) (schemes of through-computation).

Henceforth the difference boundary-value problem (22)–(24) will be termed problem (II).

5. *Difference variational problem.* Let y and λ^h be the eigenfunction and the corresponding eigenvalue of problem (II). Putting $v = y$ in formula (17) and using conditions (23) we find

$$\lambda^h = D_N[y]/H_N[y], \tag{25}$$

where

$$D_N[y] = [a, y_{\bar{x}}^2] + \bar{\sigma}_1 y_0^2 + \bar{\sigma}_2 y_N^2 \quad (y_{\bar{x}}^2 = (y_{\bar{x}})^2), \tag{26}$$

$$H_N[y] = [\rho, y^2], \quad \rho_0 = 0{\cdot}5r(0), \quad \rho_N = 0{\cdot}5r(1). \tag{27}$$

Using Green's formula, it is easy to see that difference boundary-value problem (II) is equivalent to the following variational problem:

(1) find the minimum of the functional $D_N[\varphi]$ in the class of mesh functions φ, satisfying the conditions

$$a_1 \varphi_{x,0} - \bar{\sigma}_1 \varphi_0 + hD_N[\varphi]\rho_0\varphi_0 = 0, \quad a_N \varphi_{\bar{x},N} + \bar{\sigma}_2 \varphi_N - hD_N[\varphi]\rho_N\varphi_N = 0,$$
$$H_N[\varphi] = 1;$$

and

$$\lambda_1^h = D_N[y_1] = \min D_N[\varphi]$$

is the least eigenvalue and y_1 the corresponding eigenfunction of problem (II);

(2) find the minimum of $D_N[\varphi]$ in the class of comparison functions satisfying the conditions

$$H_N[\varphi] = 1, \quad H_N[\varphi, y_m] = [\rho\varphi, y_m] = 0 \qquad \text{for } m = 1, 2, \ldots, n-1,$$
$$a_1 \varphi_{x,0} - \bar{\sigma}_1 \varphi_0 + hD_N[\varphi]\rho_0\varphi_0 = 0, \quad a_N \varphi_{\bar{x},N} + \bar{\sigma}_2 \varphi_N - hD_N[\varphi]\rho_N\varphi_N = 0,$$

where y is the eigenfunction of the number m: moreover

$$\min D_N[\varphi] = D_N[y_n] = \lambda_n^h$$

is the nth eigenvalue and $y_n(x)$ its eigenfunction.

6. *Integral relation.* Let us examine Green's difference function $G^h(x, \xi)$, which is definable from the conditions:

$$(aG_{\bar{x}}^h)_x - dG^h = -\frac{\delta(x, \xi)}{h}, \qquad \text{where } \delta(x, \xi) = \begin{cases} 1, & x = \xi, \\ 0, & x \neq \xi, \end{cases}$$

$$a_1 G_{x,0}^h - \bar{\sigma}_1 G_0^h = 0, \qquad a_N G_{\bar{x},N}^h + \bar{\sigma}_2 G_N^h = 0.$$

Introducing the functions α^h and β^h which are solutions of the homogeneous equation

$$(ay_{\bar{x}})_x - dy = 0,$$

satisfying the initial conditions

$$a_1 \alpha_{x,0}^h = 1, \quad a_1 \alpha_{x,0}^h - \bar{\sigma}_1 \alpha_0^h = 0; \quad a_N \beta_{\bar{x},N}^h = -1, \quad a_N \beta_{\bar{x},N}^h + \bar{\sigma}_2 \beta_N^h = 0,$$

we represent Green's function, by analogy with [4], in the form

$$G^h(x, \xi) = \begin{cases} \dfrac{1}{\Delta} \alpha^h(x) \beta^h(\xi), & x < \xi, \\[2ex] \dfrac{1}{\Delta} \alpha^h(\xi) \beta^h(x), & x > \xi, \end{cases} \tag{28}$$

where

$$\Delta = \alpha_N^h + \frac{1}{\bar{\sigma}_2}[1+(d, \alpha^h)] = \beta_0^h + \frac{1}{\bar{\sigma}_1}[1+(d, \beta^h)] = \text{const.}$$

From this there immediately follows the symmetry of Green's function

$$G^h(x, \xi) = G^h(\xi, x).$$

By analogy with [4] it is easy to show that G^h and its difference derivatives $G^h_{\bar{x}}$ and $G^h_{\bar{\xi}}$ are bounded:

$$0 \leqslant G^h < M_1, \quad \| G^h_{\bar{x}} \|_0 \leqslant M_2, \| G^h_{\bar{\xi}} \|_0 \leqslant M_2,$$

where M_1 and M_2 are positive constants independent of h.

If $d \equiv 0$, then α^h and β^h have the form

$$\alpha^h(x) = \frac{1}{\bar{\sigma}_1} + \sum_{x'=h}^{x'=x} \frac{h}{a(x')}, \qquad \beta^h(x) = \frac{1}{\bar{\sigma}_2} + \sum_{x'=x+h}^{x'=1} \frac{h}{a(x')},$$

$$\Delta = \frac{1}{\bar{\sigma}_1} + \frac{1}{\bar{\sigma}_2} + (1, 1/a].$$

If, moreover, $\bar{\sigma}_1 = 0$, then $\alpha^h(x) \equiv 1$, $\Delta \equiv 1$

From now on we shall require

LEMMA 1. *Let $G_0^h(x, \xi)$ be Green's function of the difference operator $(ay_{\bar{x}})_x$ with boundary conditions (23), and $G_0(x, \xi)$ Green's function of the differential operator $(ku')'$ with conditions (2). If $k(x) \in Q^0$ and $\|a-k\|_1 = \rho(h)$, then $\| G_0^h - G_0 \|_0 = \rho(h)$, where $\rho(h) \to 0$ when $h \to 0$.*

In fact, noting that

$$\| \alpha^h - \alpha^0 \|_0 = \rho(h), \quad \| \beta^h - \beta^0 \|_0 = \rho(h),$$

where

$$\alpha^0(x) = \frac{1}{\sigma_1} + \int_0^x \frac{dt}{k(t)}, \qquad \beta^0(x) = \frac{1}{\sigma_2} + \int_x^1 \frac{dt}{k(t)},$$

we find

$$\| G_0^h - G_0 \|_0 \leqslant M \| a-k \|_1 = \rho(h),$$

since G_0 is defined by the same formula (23), in which α^h and β^h are replaced by the functions α^0 and β^0.

Using the second Green's formula (18) it is easy to see that the Sturm–Liouville difference problem is equivalent to the difference analogue of the integral equation

$$y = \lambda^h [G^h, \rho y], \tag{29}$$

which, with the aid of the substitution

$$\varphi(x) = \sqrt{\rho(x)}\, y(x), \qquad K^h(x, \xi) = \sqrt{\rho(x)\rho(\xi)}\, G^h(x, \xi)$$

is reduced to the equation

$$\varphi = \lambda^h [K^h, \varphi] \tag{30}$$

with symmetrical kernel $K^h(x, \xi)$.

If we use Green's function G_0^h of the operator $L_h^{(k)}$, instead of (30) we obtain the equation

$$y = [G_0^h, (\lambda^h \rho - d) y], \tag{31}$$

where $d_0 = 0{\cdot}5q(0)$, $\quad d_N = 0{\cdot}5q(1)$, $\quad \rho_0 = 0{\cdot}5r(0)$, $\quad \rho_N = 0{\cdot}5r(1)$.

7. *Properties of eigenfunctions and eigenvalues.* The Sturm–Liouville difference problem (II) is a purely algebraic problem. It is therefore not difficult to prove the following propositions:

(1) there exist N real eigenvalues $\lambda_1^h, \lambda_2^h, \ldots, \lambda_N^h$, to which there correspond the eigenfunctions $y_1, y_2, \ldots, y_N$;

(2) to each eigenvalue there corresponds only one eigenfunction (this is proved in the same way as for problem (I) in Section 1), so that we can write $\lambda_1^h < \lambda_2^h \ldots < \lambda_n^h < \ldots < \lambda_N^h$;

(3) all eigenvalues are positive (this follows from (25));

(4) eigenfunctions form an orthogonal system normalized with weight ρ

$$H_N[y_m, y_n] = [\rho y_m, y_n] = \begin{cases} 0, & m \neq n, \\ 1, & m = n; \end{cases}$$

(5) $$M_1 n^2 \leqslant \lambda_n^h \leqslant M_2 n^2 \quad (n = 1, 2, \ldots, N), \tag{32}$$

where M_1 and M_2 are positive constants independent of both h and n.

LEMMA 2. *Let y_n, λ_n^h be the n-th normalized eigenfunction and the n-th eigenvalue of problem* (II). *Then the functions y_n and $(y_n)_{\bar{x}}$ are uniformly bounded*:

$$\|y_n\|_0 \leqslant M_1 n^{\frac{1}{2}}, \quad \|(y_n)_{\bar{x}}\|_0 \leqslant M_2 n^{\frac{3}{2}} \tag{33}$$

where M_1 and M_2 are constants independent of both h and n.

Let x and x' be any two points of the mesh ω_h. Let us examine the two obvious identities:

$$y^2(x) - y^2(x') = \sum_{s=x'+h}^{s=x} \left(y^2(s)\right)_{\bar{x}} \cdot h = \sum_{s=x'+h}^{s=x} [y(s) + y(s-h)] y_{\bar{x}}(s) \cdot h, \tag{34}$$

$$\begin{aligned} \left(a(x) y_{\bar{x}}(x)\right)^2 - \left(a(x') y_{\bar{x}}(x')\right)^2 &= \sum_{s=x'}^{s=x-h} h\left[\left(a(s) y_{\bar{x}}(s)\right)^2\right]_x \\ &= \sum_{s=x'}^{s=x-h} \left(a(s) y_{\bar{x}}(s)\right)_x \cdot [a(s) y_{\bar{x}}(s) + a(s+h) y_x(s)] \cdot h \\ &= \sum_{s=x'}^{s=x-h} \left(d(s) - \lambda^h \rho(s)\right)[a(s) y_{\bar{x}}(s) + a(s+h) y_x(s)] y(s) \cdot h \\ &\qquad (x > 0, \quad x' > 0) \end{aligned} \tag{35}$$

(the index n is dropped for the time being).

From the condition of normalization $[\rho, y^2] = 1$ it follows that there exists at least one point x', at which $\rho(x') y^2(x') \leqslant 1$ and hence, $y^2(x') \leqslant 1/c_3$. Using the Cauchy–Bunyakovskii inequality for transformation of the right-hand side of (34) and bearing (32) in mind, we shall obtain

$$y^2(x) \leqslant \frac{1}{c_3} + \frac{2}{\sqrt{c_1 c_3}} [\rho, y^2]^{\frac{1}{2}} \cdot (a, y_{\bar{x}}^2]^{\frac{1}{2}} \leqslant \frac{1}{c_3} + \frac{2\sqrt{\lambda_n^h}}{\sqrt{c_1 c_3}} \leqslant M_1^2 n. \tag{36}$$

Further from the condition $(a, y_{\bar{x}}^2] \leqslant \lambda^h$ it follows that there exists a point x', at which

$$a(x')\, y_{\bar{x}}^2(x') \leqslant \lambda^h,$$

i.e.

$$(a(x')\, y_{\bar{x}}(x'))^2 \leqslant c_2 \lambda^h.$$

Then, by using the Cauchy–Bunyakovskii inequality for the transformation of the right-hand side of (35) and bearing in mind (25) and (32), we shall have

$$y_{\bar{x}}^2(x) \leqslant \frac{c_2}{c_1^2}\lambda^h + 2\sqrt{\frac{c_5}{c_3}\frac{c_5}{c_1^2}}(\lambda^h)^{1/2} + \frac{2\sqrt{c_2 c_4}}{c_1^2}(\lambda^h)^{3/2} \leqslant M_2^2 n^3. \tag{37}$$

From the inequalities (36) and (37), by virtue of the arbitrary nature of x, the estimates (33) follow.

Estimates (33) are rougher in comparison with estimates (7) for problem (I). However, so as not to complicate the exposition, we shall not deal with their refinement, all the more since for our purposes there is no need to do so.

The condition of normalization $H_N[y] = 1$ determines the eigenfunction y with accuracy except for the sign. For unambiguous definition of the eigenfunction we must introduce a further condition to select the sign. In the case of a boundary condition when $x = 0$ of the second or third kind, we may require for this purpose that $y(0) > 0$, and in the case of a boundary condition of the first kind, $(y(0) = 0)$, require that $y_{x,0} > 0$. An analogous selection of the sign may also be made for the eigenfunctions $u(x)$ of the initial problem (I). Henceforward normalization of the eigenfunctions, side by side with the conditions $H_N[y] = 1$ and $H[u] = 1$, will also include selection of the sign by the method indicated above.

§ 2. CONVERGENCE OF SOLUTIONS OF THE DIFFERENCE PROBLEM

The convergence of the eigenvalues and eigenfunctions of the difference problem (II) to the eigenvalues and functions of the initial Sturm–Lioville problem (I) when $N \to \infty$ $(h \to 0)$ was proved by Courant [1] for the simplest scheme $a = k(x-h)$, $d = q(x)$, $\rho = r(x)$ in the class of smooth coefficients. In this section, we shall use Courant's method to prove convergence for problem (II) in the class of piecewise-continuous coefficients $(k, q, r \in Q^{(0)})$.

We shall prove the following

THEOREM 1. *If the scheme $L_h^{(k,q,\lambda r)}$ has zero rank, then the solution $(\lambda_r^h, y_n(x))$ of problem* (II) *converges on any sequence of meshes when $h \to 0$ to the corresponding solution $(\lambda_n, u_n(x))$ of problem* (I)

$$\lambda_n^h - \lambda_n = \rho(h), \quad \|y_n - u_n\|_0 = \rho(h)$$

for any piecewise-continuous coefficients $k, q, r \in Q^{(0)}$, satisfying condition (3).

We shall consider the case of the first eigenvalue $(n = 1)$.

Let $\varphi(x)$ be any piecewise-smooth function. It is not difficult to see that

$$\lim_{N\to\infty} D_N[\varphi] = D[\varphi], \qquad \lim_{N\to\infty} H_N[\varphi] = H[\varphi].$$

It follows from this, that $D_N[\varphi] \leqslant M$, where M is a constant independent of N (or $h = 1/N$).

Let $y = y(x, h)$ be a mesh function which realizes the minimum of the functional $D_N[\varphi]$:

$$\lambda^h = D_N[y]$$

with the condition of normalization $H_N[y] = 1$.

We shall consider the sequence of mesh functions $\{y(x, h)\}$ on some sequence of meshes.

LEMMA 3. *The sequence of functions $\{y(x, h)\}$ is equally continuous and uniformly bounded.*

If x', x'' are mesh points, then

$$y(x'', h) - y(x', h) = \sum_{s=x'}^{s=x''-h} h \cdot y_x(s, h).$$

Then, using the Cauchy–Bunyakovskii inequality and the limitation on $D_N[y]$ we obtain

$$|y(x'', h) - y(x', h)| \leqslant \sqrt{(1, y_{\bar{x}}^2]} \cdot \sqrt{|x'' - x'|} \leqslant M\sqrt{|x'' - x'|}. \tag{38}$$

From the condition of normalization $H_N[y] = 1$ it follows that at least at one point $x = x'$ the inequality

$$\rho(x')y^2(x', h) \leqslant 1, \qquad \text{i. e.} \qquad |y(x', h)| \leqslant 1/\sqrt{c_3}.$$

is satisfied.

From this, and from (38) there follows the uniform bounding of the sequence $\{y(x, h)\}$:

$$|y(x'', h)| \leqslant |y(x'', h) - y(x', h)| + |y(x', h)| \leqslant M.$$

According to Artzel's theorem, which is used for the sequence of mesh functions, there exists some sub-sequence $\{y(x, h_k)\}$ which converges uniformly to some function $\tilde{u}(x)$, which is continuous on the line segment $[0, 1]$:

$$\|y(x, h_k) - \tilde{u}(x)\|_0 = \rho(h_k).$$

We shall assume that the sequence $\{h_k\}$ is such that the numerical sequence $\{\lambda^{h_k} = \lambda(h_k)\}$ converges to some limit $\tilde{\lambda}$:

$$\lim_{h_k\to 0} \lambda(h_k) = \tilde{\lambda}.$$

Otherwise we would have selected from it a convergent subsequence and would have considered only those numbers k which correspond to this subsequence.

LEMMA 4. *For some sequence*

$$\lim_{h_k\to 0} \lambda(h_k) = \tilde{\lambda},$$

then $\tilde{\lambda} \leqslant \lambda$.

Let $u^*(x)$ be some piecewise-smooth function, for which

$$\lambda^* = D[u^*]/H[u^*] \leqslant \lambda + \varepsilon, \qquad \varepsilon > 0,$$

and let

$$\lambda^*(h_k) = D_{N_k}[u^*]/H_{N_k}[u^*] \qquad (N_k = 1/h_k).$$

By virtue of the principle of the minimum $\lambda(h_k) \leqslant \lambda^*(h_k)$ and $\lambda^*(h_k) \to \lambda^*$ when $h_k \to 0$. In the limit when $h_k \to 0$, we obtain

$$\tilde{\lambda} \leqslant \lambda^* \leqslant \lambda + \varepsilon.$$

From this, by virtue of the arbitrary nature of ε, it follows that $\tilde{\lambda} \leqslant \lambda$.

Our immediate aim is to show that the limit function satisfies conditions (1) and (2) when $\lambda = \tilde{\lambda}$.

As was shown in § 1, Section 6, difference problem (II) is equivalent to the equation

$$y = [G_0^h, (\lambda^h \rho - d) y], \tag{31}$$

where G_0^h is Green's function for the operator $(ay_{\bar{x}})_x$ with conditions (23). We shall now perform the limit transition to (31) when $h_k \to 0$ and use Lemma 1. We shall then obtain

$$\tilde{u}(x) = \int_0^1 G_0(x, \xi)\left(\tilde{\lambda} r(\xi) - q(\xi)\right)\tilde{u}(\xi)\, d\xi, \tag{39}$$

where $G_0(x, \xi)$ is Green's function for the operator $(ku')'$ with conditions (2).

From this, by the definition of Green's function, it follows that the solution $\tilde{u}(x)$ of the integral equation (39) satisfies the differential equation

$$L^{(k,q)}\tilde{u} + \tilde{\lambda} r \tilde{u} = 0$$

and the boundary conditions

$$k(0)\tilde{u}'(0) - \sigma_1 \tilde{u}(0) = 0, \qquad k(1)\tilde{u}'(1) + \sigma_2 \tilde{u}(1) = 0.$$

Since to each eigenvalue of problem (I) there corresponds only one eigenfunction $u_1(x)$, and $\lambda = \lambda_1$ is the least eigenvalue, then

$$\tilde{\lambda} = \lambda_1 \quad \text{and} \quad \tilde{u}(x) \equiv u_1(x).$$

It also follows from what has been said that the whole sequence $\{y(x, h)\}$ converges uniformly to $u(x)$ and $\lambda_1^h = \lambda_1(h)$ converges to λ_1 when $h \to 0$:

$$\|y_1 - u_1\|_0 = \rho(h), \qquad \lambda_1^h - \lambda_1 = \rho(h).$$

The reasoning above related to the least eigenvalue λ_1^h.

In the case of other eigenvalues λ_n^h for $n > 1$ all the reasoning remains valid, if we bear in mind that λ_n^h and λ_n are defined as the minimums of the functionals $D_N[\varphi]$ and, correspondingly, $D[\varphi]$, with the further conditions of the orthogonality of $H_N[\varphi, y_m] = 0$ and $H[\varphi\, u_m] = 0$ $(1 \leqslant m < n)$. Theorem 1 is thus proved.

§ 3. ON THE ACCURACY OF THE DIFFERENCE METHOD

1. *An equation for eigenfunction error.* Let (λ^h, y) be the solution of the difference problem (II) and $(\lambda\, u,)$ the corresponding solution of the initial Sturm–Liouville problem (I). We shall answer the question of the asymptotic order when $h \to 0$ of the error $z = y - u$ in the norm $\|\ \|_0$ and also for the difference $\Delta\lambda = \lambda^h - \lambda$. We shall first of all formulate the boundary condition for z. We shall substitute $y = z + u$ in equation (22) and bear in mind equation (1) for u; we shall then obtain for z the inhomogeneous difference equation

$$(az_{\bar{x}})_x - d \cdot z + \lambda^h \rho \cdot z = -\Psi, \tag{40}$$

where

$$\Psi = \psi + (\lambda^h - \lambda)\, \rho u, \tag{41}$$

$$\psi = L_h^{(k,\, q,\, \lambda r)} u - L^{(k,\, q,\, \lambda r)} u = [(au_{\bar{x}})_x - (ku')'] - (d - q)\, u + \lambda\, (\rho - r)\, u. \tag{42}$$

Function ψ is the error of approximation of the difference scheme on the solution of equation (1).

For function z we obtain the inhomogeneous boundary conditions

$$a_1 z_{x,0} - \bar{\sigma}_1 z_0 + h\lambda^h \rho_0 z_0 = \nu_1 \qquad a_N z_{\bar{x},N} + \bar{\sigma}_2 z_0 - h\lambda^h \rho_N z_N = -\nu_2, \tag{43}$$

where

$$\nu_1 = h\,(\lambda^h - \lambda)\, \rho_0 u_0 + \bar{\nu}_1, \qquad \nu_2 = h\,(\lambda^h - \lambda)\, \rho_N u_N + \bar{\nu}_2 \qquad (\rho_0 = 0{\cdot}5r\,(0)), \quad \rho_N = 0{\cdot}5r\,(1)), \tag{44}$$

$$\bar{\nu}_1 = \big(a_1 u_x(0) - \bar{\sigma}_1 u(0) + h\lambda\rho_0 u(0)\big) - \big(k(0)\, u'(0) - \sigma_1 u(0)\big), \tag{45}$$

$$\bar{\nu}_2 = \big(a_N u_{\bar{x}}(1) + \bar{\sigma}_2 u(1) - h\lambda\rho_N u(1)\big) - \big(k(1)\, u'(1) + \sigma_2 u(1)\big). \tag{46}$$

Thus to ascertain the accuracy of the solution of the difference problem we have to estimate the solution of equation (40) with boundary conditions (43). This problem will be called problem (III).

2. *Formula for* $\Delta\lambda = \lambda^h - \lambda$. Parameter λ^h is an eigenvalue. Therefore problem (III) is solvable only in the case when the eigenfunction y of problem (II) is orthogonal to the right-hand side of the equation and the boundary conditions, or more accurately:

$$[\Psi, y] = [\psi, y] + (\lambda^h - \lambda)\, [\rho u, y] = 0, \tag{47}$$

where

$$[\psi, y] = (\psi, y) + \bar{\nu}_1 y_0 + \bar{\nu}_2 y_N,$$

$$h\psi_0 = \bar{\nu}_1, \qquad h\psi_N = \bar{\nu}_2, \qquad \rho_0 = 0{\cdot}5r\,(0), \qquad \rho_N = 0{\cdot}5r\,(1).$$

We shall assume u and y to be normalized functions, such that

$$H[u] = 1, \qquad H_N[y] = 1.$$

By virtue of Theorem 1 $\|y - u\|_0 \to 0$ when $h \to 0$. Therefore with a sufficiently small $h \leqslant h_0$ we can assert that $[\rho u, y] \neq 0$. To the eigenvalue λ^h there corresponds only one eigenfunction, which is definable with accuracy up to the arbitrary factor.

We shall select the factor C in such a way that the function $\bar{y} = Cy$ is orthogonal to the difference $\bar{z} = \bar{y} - u$:

$$[\rho\bar{y}, \bar{z}] = 0, \qquad \text{where} \qquad \bar{z} = \bar{y} - u. \tag{48}$$

It follows from this that

$$[\rho, yu] = [\rho, \bar{y}y] = C[\rho, y^2] = C,$$
$$[\rho, u^2] = [\rho, \bar{y}^2] + [\rho, \bar{z}^2] = C^2 + [\rho, \bar{z}u],$$
$$C^2 = [\rho, u^2] - [\rho, \bar{z}u]$$

or

$$1 - C^2 = [\rho, \bar{z}u] - (H_N[u] - H[u]). \tag{49}$$

By virtue of Theorem 1 it is clear that $C^2 \to 1$ when $h \to 0$. We shall assume that $C > 0$, i.e., the signs of u and y are coordinated (see § 1, Section 7). Formula (49) will be needed for the estimate of $|C^2 - 1|$.

We shall use condition (47) to define

$$\Delta\lambda = \lambda^h - \lambda = -[\psi, y]/[\rho u, y] = -[\psi, \bar{y}]/C^2. \tag{50}$$

We shall transform the right-hand side. We introduce the function η with the aid of the conditions

$$\eta_{\bar{x}} = \psi, \qquad \eta_0 = h\psi_0 = \bar{\nu}_1, \qquad \eta(x) = \sum_{x'=0}^{x'=x} h\,\psi(x').$$

Applying summation to both sides of (16)

$$[\psi, \bar{y}] = -[\eta, y_x) + y_N \eta_N,$$

and also Lemma 2, we obtain

$$|[\psi, \bar{y}]| \leqslant M(n)\|\psi\|_4,$$

where

$$\|\psi\|_4 = \sum_{i=1}^{N-1} h\left|\sum_{k=1}^{i} h\psi_k\right| + \left|\sum_{i=0}^{N} h\psi_i\right| + h|\psi_0| + h|\psi_N|; \tag{14}$$

$M(n)$ is a positive constant dependent on the number n of the eigenvalue.

This proves

LEMMA 5. *If the conditions of Theorem 1 are satisfied, then*

$$|\lambda_n^h - \lambda_n| \leqslant M(n)\|\psi\|_4, \tag{51}$$

where the function ψ is defined by formula (42).

3. *Integral relation for $\bar{z} = \bar{y} - u$.* For estimating $\bar{z}$ we shall reduce problem (III) to the "integral" equation

$$\bar{z} = \lambda^h[G, \rho\bar{z}] + [G, \Psi], \tag{52}$$

where $G = G^h(x, \xi)$ is Green's difference function of the operator $L_h^{(k,q)}y = (ay_{\bar{x}})_x - dy$ with boundary conditions (23).

The eigenfunction $\bar{y}$ of problem (II) satisfies the equation

$$\bar{y} = \lambda^h[G, \rho\bar{y}]. \tag{29'}$$

We introduce the symmetrical kernel

$$K(x,\xi) = \sqrt{\rho(x)\,\rho(\xi)}\,G(x,\xi) \qquad (x \in \omega_h, \quad \xi \in \omega_h)$$

(the dependence of G and K on h will not be indicated) and the new functions

$$v(x) = \sqrt{\rho(x)}\,\bar{z}(x), \qquad \varphi(x) = \sqrt{\rho(x)}\,y(x) \qquad (\rho \geqslant 0{\cdot}5c_3 > 0).$$

Then from (52) and (29′) we shall obtain for the mesh functions $v(x)$ and $\varphi(x)$ the equations

$$v = \lambda^h [K, v] + f, \quad f = [K, \overline{\Psi}], \quad \overline{\Psi} = \Psi/\sqrt{\rho}, \tag{53}$$

$$\varphi = \lambda^h [K, \varphi]. \tag{54}$$

It is easy to see that the condition of the orthogonality of φ to the right-hand side of f is automatically satisfied by virtue of condition (47) and equation (54)

$$[\varphi, f] = [\varphi(x), [K(x,\xi), \overline{\Psi}(\xi)]] = [\overline{\Psi}(\xi), [K(\xi, x), \varphi(x)]]$$
$$= \frac{1}{\lambda^h}[\overline{\Psi}(\xi), \varphi(\xi)] = \frac{1}{\lambda^h}\left[\frac{\Psi}{\sqrt{\rho}}, y\sqrt{\rho}\right] = \frac{1}{\lambda^h}[\Psi, y] = 0$$

Let $\lambda^h = \lambda_n^h$ be the eigenvalue of number n and $\varphi_n(x)$ its normalized eigenfunction ($[\varphi_n, \varphi_n] = 1$).

Condition $[y, \rho\bar{z}] = 0$ is written in the form

$$[\varphi_n, v] = 0. \tag{55}$$

Moreover, we have

$$[\varphi_n, f] = 0. \tag{56}$$

We shall find the resolvent $R(x, \xi; \lambda_n^h)$, with the aid of which the solution of equation (53) is given by the formula

$$v = f + \lambda_n^h [R, f]. \tag{57}$$

From this and from conditions (55) and (56) there follows the orthogonality of R to φ_n:

$$[R, \varphi_n] = 0. \tag{58}$$

This same property is possessed by the kernel

$$K_1(x,\xi) = K(x,\xi) - \frac{\varphi_n(x)\,\varphi_n(\xi)}{\lambda_n^h} = \sum_{\substack{k=1\\(k\neq n)}}^{N} \frac{\varphi_k(x)\,\varphi_k(\xi)}{\lambda_k^h}.$$

The resolvent R is determined from the equation

$$R(x,\xi;\lambda_n^h) = K_1(x,\xi) + \lambda_n^h [K_1(x,t), R(t,\xi;\lambda_n^h)]$$

and can be written in the form

$$R = K_1 + R_1,$$

where

$$R_1 = R_1(x,\xi;\lambda_n^h) = \sum_{\substack{k=1\\(k\neq n)}}^{N} \frac{\varphi_k(x)\,\varphi_k(\xi)}{\lambda_k^h(\lambda_k^h/\lambda_n^h - 1)}.$$

Kernel K_1 is bounded by virtue of the limitation of Green's function and the eigenfunction $\varphi_n(x)$ (Lemma 2). We shall therefore have:

$$\sum_{\substack{k=1\\(k\neq n)}}^{N} \frac{\varphi_k^2(x)}{(\lambda_k^h)^2} \leqslant [K_1^2(x,\xi), 1] \leqslant M,$$

$$\| R_1 \|_2^2 = [R_1^2(x,\xi;\lambda_n^h), 1] = \sum_{\substack{k-1\\(n\neq k)}}^{N} \frac{\varphi_k^2(x)}{(\lambda_k^h)^2(1-\lambda_k^h/\lambda_n^h)^2} \leqslant M,$$

$$\| R \|_2 \leqslant \| K_1 \|_2 + \| R_1 \|_2 \leqslant M.$$

Resorting to formula (57), we obtain

$$\| v \|_0 \leqslant (1+\lambda_n^h \| R \|_2) \| f \|_0. \tag{59}$$

4. *A priori estimates*

THEOREM 2. *Let* (λ_n^h, y_n) *be the eigenvalue and the normalized eigenfunction of number* n *of problem* (II), *and* (λ_n, u_n) *the eigenvalue and normalized eigenfunction of number* n *of problem* (I).

If the conditions of Theorem 1 are satisfied, then at a sufficiently small $h \leqslant h_0$ *the inequalities*

$$| \lambda_n^h - \lambda_n | \leqslant M_1(n) \| \psi \|_4, \tag{51}$$

$$\| y_n - u_n \|_0 \leqslant M_2(n) \| \psi \|_4 + M \,| H_N[u_n] - H[u] |, \tag{60}$$

are satisfied, where $M_1(n)$ *and* $M_2(n)$ *are constants dependent on* n *and independent of* h.

For proof of the theorem it is sufficient to establish inequality (60), since an estimate for $\lambda_n^h - \lambda_n$ has already been obtained in Section 2 (Lemma 5).

We shall turn to inequality (59) and examine $\|f\|_0$, where,

$$f = [K, \bar{\psi}] + (\lambda^h - \lambda)[K, \sqrt{\rho}u], \quad \bar{\psi} = \psi/\sqrt{\rho}.$$

The second addend, taken with respect to the norm $\| \ \|_0$, is dominated by the quantity $M\ |\lambda^h - \lambda|$, or, in accordance with Lemma 5, the quantity $M_1(n) \| \psi \|_4$. We shall now transform the expression

$$[K, \bar{\psi}] = \sqrt{\rho(x)}\, [G(x,\xi), \psi(\xi)].$$

Introducing the function η, with the aid of the conditions

$$\eta_{\bar{x}} = \psi, \quad \eta_0 = h\bar{\psi}_0 = \bar{\nu}_1,$$

we obtain

$$[K, \bar{\psi}] = \sqrt{\rho(x)}\, \{-[G_\xi(x,\xi), \eta(\xi)) + G(x,1)\psi_N h - G(x,1)\eta_{N-1} h\}.$$

From this, by virtue of the boundedness of Green's function and its first difference derivatives (see § 1, Section 6) there follows

$$\| [K, \bar{\psi}] \|_0 \leqslant M \| \psi \|_4.$$

Thus

$$\| f \|_0 \leqslant M(n) \| \psi \|_4$$

and hence

$$\|\bar{z}\|_0 \leqslant \frac{M(n)}{\sqrt{c_3}}(1+\lambda_n^h\|R\|_2)\|\psi\|_4 \leqslant M(n)\|\psi\|_4. \tag{61}$$

We are interested in the difference

$$z = y - u,$$

which is expressed by $\bar{z}$:

$$z = \frac{\bar{z}}{C} + \frac{1-C}{C}u,$$

$$\|z\|_0 \leqslant \frac{1}{C}\|\bar{z}\|_0 + \frac{1-C^2}{C(1+C)}\|u\|_0 \leqslant M(\|\bar{z}\|_0 + |1-C^2|) \qquad \text{when } h \leqslant h_0,$$

since $C \to 1$ when $h \to 0$, and $\|u\|_0$ is bounded by virtue of (7).

Turning now to formula (49), we find

$$|1-C^2| \leqslant M\|\bar{z}\|_0 + |H_N[u_n] - H[u]|$$

and hence

$$\|z\|_0 \leqslant M(n)\|\bar{z}\|_0 + M|H_N[u_n] - H[u]|.$$

Then, bearing in mind estimate (61) for $\|\bar{z}\|_0$ we obtain inequality (60). Theorem 2 is proved.

5. *The order of accuracy in the class of smooth coefficients.* By virtue of Theorem 2 the order of accuracy of the solution of difference problem (II) depends on the error of approximation of the difference scheme, including the boundary conditions, and also on the error of approximation of the normalized functional H_N, i.e., on the magnitude

$$\chi = H_N[u_n] - H[u_n].$$

The estimation of ψ with respect to the norm $\|\psi\|_4$ proves useful even in the class $C^{(m)}$, since it enables us to reduce by one order the requirement of differentiability of the function $k(x)$, and also the rank of the standard functional $A[\bar{k}(s)]$. In fact, as *a priori* estimates (51) and (60) show, for the difference scheme to have an mth ($m = 1, 2$) order of accuracy, it is sufficient for the condition $\|\psi\|_4 = O(h^m)$ to be satisfied. At the same time the scheme may also fail to have an mth order of approximation, i.e. the condition $\psi = O(h^m)$ will not be satisfied.

It will be shown below that $\|\psi\|_4 = O(h^m)$ in the class $k, q, r \in C^{(m-1,1)}$, if the scheme has rank m and satisfies conditions (20) for an mth order of approximation.

In § 1, Section 3, we agreed to consider only schemes of the standard type. We would also recall that $D[f]$ and $R[f]$ are linear functionals.

THEOREM 3. *If the difference scheme $L_h^{(k,q,\lambda r)}$ has 2nd rank and satisfies the conditions* (21) *necessary for the second order of approximation, then the solution of problem* (II) *for $k, q, r \in C^{(1,1)}$ has a second order of accuracy*:

$$|\lambda_n^h - \lambda_n| \leqslant M(n)h^2, \qquad \|y_n - u_n\|_0 \leqslant M(n)h^2.$$

For proof of the theorem it is sufficient to estimate $\|\psi\|_4$ and χ and use Theorem 2, the conditions of which are satisfied.

We shall consider the error of approximation

$$\psi = \varphi - (d-q)u + \lambda(\rho - r)u$$

(the index n is dropped), where

$$\varphi = (au_{\bar{x}})_x - (ku')' = L_h^{(k)}u - L^{(k)}u.$$

If the conditions of the theorem are satisfied, there exist derivatives u'' and $(ku')''$, satisfying the Lipschitz condition.

We shall therefore have

$$(ku')' = (\overline{ku'})_x + O(h^2),$$

where the bar above denotes that the expression is taken at the point $\bar{x} = x - 0{\cdot}5h$.

Let us first consider

$$\sum_{x'=h}^{x'=x-h} h\varphi(x') = \sum_{x'=h}^{x'=x-h} \left(a(x')u_{\bar{x}}(x') - \overline{k(x')u'(x')}\right)_x h + O(h^2)$$

$$= a(x)u_{\bar{x}}(x) - k(\bar{x})u'(\bar{x}) - \left(a(x')u_{\bar{x}}(x') - k(\bar{x}')\bar{u}'(\bar{x}')\right)\big|_{x'=h} + O(h^2).$$

By virtue of conditions (21) $a = \bar{k} + O(h^2)$. Then, bearing in mind that $u_{\bar{x}} = \bar{u}' + O(h^2)$ (since u'' satisfies the Lipschitz condition), we obtain $au_{\bar{x}}\,\overline{k\bar{u}'} = O(h^2)$, and hence

$$\sum_{x'=h}^{x'=x-h} h\varphi(x') = O(h^2), \qquad \|\varphi\|_3 = O(h^2).$$

The difference boundary conditions (23), as we saw in § 1, Section 4, have a second order of approximation in the class of solutions of equation (1), i.e., $\bar{\nu}_1 = O(h^2)$ and $\bar{\nu}_2 = O(h^2)$. It follows from this that

$$\|\varphi\|_4 = O(h^2), \quad \text{more accurately,} \quad \|\varphi\|_4 \leqslant M \cdot h^2.$$

Then, bearing in mind that by virtue of conditions (20) $d - q = O(h^2)$, $\rho - r = O(h^2)$, we shall have

$$\|\psi\|_4 \leqslant M(n)h^2.$$

Noting that $H_N[u]$ is a quadrature formula for $H\,[u]$ of the second order of accuracy in the case $r \in C^{(1,1)}$ and $u \in C^{(1,1)}$ we find

$$\chi = O(h^2).$$

Theorem 3 is proved.

6. *The order of accuracy in the class of discontinuous coefficients.* We shall now assume that $k, q, r \in Q^{(1,1)}[0,1]$; then $u' \in Q^{(1,1)}$, $(ku')' \in Q^{(1,1)}$. We shall denote $\xi_j = x_{n_j} + \theta_j h (x_{n_j} = hn_j, 0 \leqslant \theta_j \leqslant 1, 0 < \xi_j < 1)$ all the points of discontinuity of the functions $k(x)$, $q(x)$ and $r(x)$. The number j_0 of such discontinuities is finite: $j = 1, 2, \ldots, j_0$.

In computing ψ in this case we shall refer to § 3 of the paper [4]. Let $L_h^{(k,q,\lambda r)}$ be a scheme of 2nd rank, satisfying conditions (21). We shall represent ψ in the form of the sum

$$\psi = \bar{\psi} + \bar{\bar{\psi}},$$

where

$$\bar{\bar{\psi}} = \bar{\bar{\psi}}_i = \psi_{n_j}\delta_{i,n_j} + \psi_{n_j+1}\delta_{i,n_j+1},$$

$$j = 1, 2, \dots; j_0, \quad \delta_{i,k} = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases}.$$

By the same reasoning as in the proof of Theorem 3, we obtain

$$\|\bar{\psi}\|_4 = O(h^2), \quad \text{more accurately} \quad \|\bar{\psi}\|_4 \leqslant M(n)\, h^2. \tag{62}$$

It must be borne in mind here that $h\bar{\psi}_0 = \bar{\nu}_1, h\bar{\psi}_N = \bar{\nu}_2$.

We now turn to the computation of $\|\bar{\bar{\psi}}\|_4$. We shall note first that for every scheme of the above-mentioned family, the conditions

$$h\psi_{n_j} = O(1), \quad h\psi_{n_j+1} = O(1), \quad h(\psi_{n_j} + \psi_{n_j+1}) = O(h),$$
$$j = 1, 2, \dots, j.$$

are satisfied.

For the difference scheme

$$L_h^{(k,q,\lambda r)} y = (a y_{\bar{x}})_x - d \cdot y + \lambda \cdot \rho y \tag{63}$$

with coefficients

$$a = \left[\int_{-1}^{0} \frac{ds}{k(x+sh)}\right]^{-1}, \quad d = \int_{-0\cdot5}^{0\cdot5} q(x+sh)\, ds, \quad \rho = \int_{-0\cdot5}^{0\cdot5} r(x+sh)\, ds \tag{64}$$

the conditions [4]

$$h\psi_{n_j} = O(h), \quad h\psi_{n_j+1} = O(h), \quad h(\psi_{n_j} + \psi_{n_j+1}) = O(h^2), \quad j = 1, 2, \dots, j_0.$$

are satisfied.

It is easy to see that

$$\|\bar{\bar{\psi}}\|_3 \leqslant \sum_{j=1}^{j_0} (h^2 |\psi_{n_j}| + h|\psi_{n_j} + \psi_{n_j+1}|),$$

$$\|\bar{\bar{\psi}}\|_4 \leqslant \sum_{j=1}^{j_0} (h^2 |\psi_{n_j}| + 2h|\psi_{n_j} + \psi_{n_j+1}|).$$

LEMMA 6. *If*

$$\rho = \rho_1 = R_1[r(x+sh)] = \int_{-0\cdot5}^{0\cdot5} r(x+sh)\, ds,$$

then for $r(x) \in Q^{(0,1)}$ *and* $u \in C^{(1)}$, $u' \in Q^{(1,1)}$ *the estimate*

$$\chi = H_N[u] - H[u] = O(h^2).$$

To simplify the notation, without disrupting the generality, we can consider that there is only one point $\xi = x_n + \theta \cdot h$ of discontinuity of the function $r(x)$. We shall represent χ in the form of a sum:

$$\chi = \sum_{i=0}^{N} \Delta_i \cdot h,$$

where

$$\Delta_i = \int_{-0\cdot5}^{0\cdot5} r(x_i+sh)\,[u^2(x_i) - u^2(x_i+sh)]\,\mathrm{d}s, \qquad 0 < i < N,$$

$$\Delta_0 = 0\cdot5hr(0)\,u^2(0) - \int_0^{0\cdot5h} r(x)\,u^2(x)\,\mathrm{d}x,$$

$$\Delta_N = 0\cdot5hr(1)\,u^2(1) - \int_{1-0\cdot5h}^{1} r(x)\,u^2(x)\,\mathrm{d}x.$$

It can immediately be seen that $\Delta_0 = O(h^2)$, $\Delta_N = O(h^2)$. If $i \neq n$, $i \neq n+1$, then

$$\Delta_i = -\int_{-0\cdot5}^{0\cdot5} \big(r(x_i) + shr(x_i) + h\rho(h)\big)\big((u^2)'_i\,sh + O(h^2)\big)\,\mathrm{d}s = O(h^2).$$

We shall now assume that $0 \leqslant \theta \leqslant 0\cdot5$. Then

$$\Delta_n = \int_{-0\cdot5}^{\theta} \big(r_l + O(h)\big)\cdot O(h)\,\mathrm{d}s + \int_{\theta}^{0\cdot5} \big(r_r + O(h)\big)\cdot O(h)\,\mathrm{d}s = O(h),$$

$$r_l = r(\xi-0), \quad r_r = r(\xi+0), \quad \Delta_{n+1} = O(h^2).$$

If, however, $0\cdot5 \leqslant \theta \leqslant 1$, then

$$\Delta_n = O(h^2), \quad \Delta_{n+1} = O(h).$$

In both cases $\Delta_n + \Delta_{n+1} = O(h)$. It follows from this that

$$\chi = h(\Delta_n + \Delta_{n+1}) + O(h^2) = O(h^2).$$

COROLLARY. If $R\ [\bar{r}(s)]$ is an arbitrary functional, then

$$\chi = [\rho - \rho_l, u^2] + O(h^2) \quad \text{for} \quad r \in Q^{(0,1)}.$$

NOTE. Using the representation of linear functionals in the class of discontinuous coefficients (see § 1, Section 11, reference [4]) it can be shown that there exists only one linear canonical normalized functional, for which $\chi = O(h^2)$ in the class $Q^{(m)}$ for any $m \geqslant 1$.

We have thus proved the following theorems

THEOREM 4. *For any difference scheme of second rank, satisfying conditions* (21) *in the class* $Q^{(1,1)}$ $(k, q, r \in Q^{(1,1)})$ *the relations*

$$|\lambda_n^h - \lambda_n| \leqslant M(n)\,h, \qquad \|y_n - u_n\|_0 \leqslant M(n)\cdot h,$$

are satisfied, where $\lambda_n, u_n(x)$ *are the n-th eigenvalue and the n-th normalized eigenfunction of problem* (1) *and* λ_n^h, y_n the *n-th eigenvalue and n-th eigenfunction of the Sturm–Liouville difference problem* (II).

THEOREM 5. *The difference scheme* (63)–(64) *ensures in the class of coefficients* $k, q, r \in Q^{(1,1)}$ *the second order of accuracy*:

$$|\lambda_n^h - \lambda_n| \leqslant M(n)\, h^2, \qquad \|y_n - u_n\|_0 \leqslant M(n)\, h^2.$$

At a later date we shall consider homogeneous difference schemes, giving any order of accuracy in the class of piecewise-continuous coefficients of equation (1).

Separate consideration will also be given to the question of accuracy on non-uniform meshes.

Translated by G. K. ELLIOTT

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