HOMOGENEOUS DIFFERENCE SCHEMES*

A. N. TIKHONOV and A. A. SAMARSKII

Moscow

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ABSTRACT

Homogeneous difference schemes, suitable for transforming differential equations whose coefficients belong to certain classes of function into difference equations, are defined and discussed. The main points which arise are, first, whether the solution of the resulting difference equation converges to that of the original differential equation in the given class of coefficients, and of what order the convergence is, if it exists; and, secondly, how the "best" scheme, giving a high degree of accuracy in the widest class of coefficients and stability with respect to computing errors, can be selected. A basic lemma concerning the necessary condition for convergence is proved.

Examples are given of a difference scheme for Sturm-Liouville type operators in the class of sufficiently smooth coefficients, of a scheme for the first boundary problem in the class of smooth coefficients and in the class of discontinuous coefficients, as well as in the class of piece-wise continuous coefficients. The latter is the basic class of coefficients which is discussed in the article.

Green's function for the difference operator is constructed, and bounds are found for it and for its first difference ratios.

CONTENTS

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INTRODUCTION

THROUGH the wide-scale use of high-speed electronic computers there has arisen the need to develop homogeneous computing methods suitable for solving definite classes of mathematical problems.

The method of finite differences is often used to find approximate solutions of the differential equations $Lu = 0$ with certain supplementary conditions $lu = 0$, the unknown function u being replaced by the net function u^h defined on the difference net S_h, and the differential operator Lu by the differential operator $L_h u^h$ [1–10].

The approximate solution of the initial problem is defined as the solution of the system of difference equations $L_h u^h = 0$ with the corresponding difference conditions $l_h u^h = 0$.

Computer experience has demonstrated that it is less convenient to find a numerical method suitable for one particular problem alone, than to develop numerical algorithms suitable for solving certain classes of problem. Consider for example the class of differential equations $L^{(k)}u = 0$ with the supplementary conditions $l^{(k)}u = 0$, characterised by the type of the operators $L^{(k)}$ and $l^{(k)}$ and by the functional space K from which the vector coefficients are drawn. The difference scheme is the rule according to which the difference equations for any coefficients $k(x)$ of the functional space k are written down. The difference scheme $L_h^k u^h$, together with $l_h^k u^h$, is the computing algorithm for the solution of any problem from the given class of problems.

The following questions then arise.

(1) Does the solution of the difference problem with the difference scheme we have chosen converge to the solution of the initial problem of a differential equation in the given class of coefficients, and what is the asymptotic order of the convergence as $h \to 0$?

(2) If we think of some family of "admissible" schemes rather than of one fixed scheme, in what way should we select the "best" schemes, which give a high degree of accuracy in the widest class of coefficients and are stable with respect to disturbances caused by computing errors?

Section 1 of the Introduction is devoted to a discussion of the concept of homogeneous difference schemes. The structure of the initial classes of schemes must be studied, since the basic problem in the theory of difference schemes consists in finding a relation between the structure of a scheme and the accuracy it provides. A large amount of work concerned with particular difference schemes has been published.

We give an example in Section 2 of a difference scheme for differential operators of the Sturm-Liouville type giving high accuracy in the class of sufficiently smooth coefficients, but not even giving convergence in the class of discontinuous coefficients. As it happens, the class of equations of this type, with discontinuous coefficients, proves to be of great scientific and practical interest.

"Open" families of difference schemes are of great value, in that they enable us to solve problems with either continuous or discontinuous coefficients by a single method, without having to exclude the points of discontinuity. (It is a complicated procedure to find these points when the coefficients are obtained from the approximate solution of other equations.)

In Section 3 we give a second example which is essential in a discussion of what questions should be posed.

Section 4 summarises the results of the whole article.

This work is a revision of the results published by the authors in 1956–1960 [11–19]. While summarising them we have essentially reworked them.

1. *Homogeneous difference schemes.* The need for uniformity or homogeneity in computation, which is especially important in constructing programming cycles, puts certain requirements on the families of difference operators and schemes. First we define homogeneity of difference operators and schemes, restricting ourselves to the case of one variable; our definitions can be extended without difficulty to the case of many variables.

Let us consider some examples. For a differential operator with constant coefficients

$$Lu = \frac{d^2u}{dx^2} - qu, \quad q = \text{const}, \quad 0 < x < 1,$$

the simplest difference operator (three-point) is determined by what pattern $\mathfrak{M}_0$, consisting of the three points $m = -1, 0, 1$, and producing the functions

$$\Phi^h[\bar{u}] = \frac{\bar{u}_{-1} - 2\bar{u}_0 + \bar{u}_1}{h^2} - q\bar{u}_0 = \Phi^h(\bar{u}_{-1},\ \bar{u}_0,\ \bar{u}_1).$$

This function can be thought of as a functional given on the net function $\bar{u}$, defined on the pattern

$$\bar{u} = \{\bar{u}m\}, \qquad m \in \mathfrak{M}_0\{-1, 0, 1\}.$$

The value of the difference operator $L_h u^h$ at the point x_i of the net $S_h\{x_i = ih, h = 1/N,\ i = 0, 1, \dots, N\}$ for $1 \leqslant i \leqslant N-1$ is defined as the value of the generating function of the operator Φ^h whose arguments are the values of the net function $u^h = \{u_i^h\}$ at the points of the pattern $\mathfrak{M}_{x_i}^h$ taken to the point x_i:

$$\mathfrak{M}_{x_i}^h = \{x_i + mh\}, \qquad m \in \mathfrak{M}_0, \quad \text{i.e.}$$

$$(L_h u^h)_i = \frac{u_{i-1}^h - 2u_i^h + u_{i+1}^h}{h^2} - qu_i^h = \Phi^h[u^h(x_i + mh)].$$

Difference operators for the case of many variables are defined similarly (for example, for Laplace's equation) (see [1]).

Let us discuss the concept of the homogeneous difference operator $L_h u^h = U^h$.

First, we introduce some definitions.

A finite set of points of an integral net $\mathfrak{M}_0 = \{-m_1, -m_1 + 1, \dots, 0, 1, \dots, m_2\}$ will be called an operator pattern, and the function $\Phi^h(\bar{u}_{-m_1}, \bar{u}_{-m_1+1}, \dots, \bar{u}_0, \bar{u}_i, \dots, \bar{u}_{m_2}) = \Phi^h[\bar{u}]$ of $m_1 + m_2 + 1$ variables depending on h and on a parameter will be called the generating function (functional) of the difference operator. The function Φ^h is an h-parameter functional of the net function $\bar{u} = \{\bar{u}_m, m \in \mathfrak{M}_0\}$ given on $\mathfrak{M}_0$.

The transformation of variables

$$x = \bar{x} + s\bar{h}$$

called a shift transformation, maps the pattern $\mathfrak{M}_0$ (with $s = m$, $m \in \mathfrak{M}_0$) on to the set of points $\mathfrak{M}_{\bar{x}}^{\bar{h}}$, which we call the transformed $\bar{h}$-pattern at the point $\bar{x}$. If $\bar{x} = x_i$, $m_1 < i < N - m_2$ is an interpolation point of the net S_h and $\bar{h} = h$, then the transformed h-pattern at the point x_i is part of the net S_h.

The net function u^h given on the base net S_h, with the help of the transformed h-pattern at the point x_i, induces the function $\bar{u}_{x_i}^h$, defined on $\mathfrak{M}_0$:

$$\bar{u}_{x_i}^h = \{u^h(x + mh), m \in \mathfrak{M}_0\}, \text{ if only } \mathfrak{M}_{x_i}^h \subset S_h, \text{ i.e. } m_1 < i < N - m_2.$$

The difference operator $L_h u^h = U^h$ is said to be homogeneous if there exists a generating function (functional) $\Phi^h[\bar{u}]$, $\bar{u} = \{\bar{u}_m, m \in \mathfrak{M}_0\}$ defined for net functions given on the pattern $\mathfrak{M}_0$, such that the values of the operator at the point x_i are equal to

$$U_i^h = \Phi^h \bar{u}_{x_i}^h - \Phi^h[u^h(x_i + mh)],$$

where $\bar{u}^h_{x_i}$ is the function induced on the pattern $\mathfrak{M}_0$ by the net function u^h using the transformed h-pattern at the point x_i. Such an operator L_h transforms the net function u^h, given on S_h, into a net function U^h, defined on the net $S'_h\{x_i, m_1 \leqslant i \leqslant N-m_2\}$ which is part of the net S_h.

Consider now some examples of difference schemes for the class of differential operators

$$L^{(k)}u = \frac{d}{dx}\left[k_1(x)\frac{\partial u}{\partial x}\right] - k_2(x)u + k_3(x),$$

where the coefficients $k(x) = \big(k_1(x),\ k_2(x),\ k_3(x)\big)$ belong to some functional space K.

For example, consider the scheme

$$(L^{(k)}u^h) = U_i^{(h,\,k)} = \frac{1}{h^2}[B_i(u^h_{i+1}-u^h_i) - A_i(u^h_i - u^h_{i-1})] - D_i u_i + F_i,$$

$$0 < i < N,$$

where

$$A_i = \left(\frac{1}{h}\int\limits_{x_{i-1}}^{x_i}\frac{dx}{k_1(x)}\right)^{-1} = \left[\int\limits_{-1}^{0}\frac{ds}{k_1(x_i+sh)}\right]^{-1},$$

$$B_i = \left[\int\limits_{0}^{1}\frac{ds}{k_1(x_i+sh)}\right]^{-1} = A_{i+1},$$

$$D_i = \frac{1}{h}\int\limits_{x_i-0\cdot 5h}^{x_i+0\cdot 5h} k_2(x)\,dx = \int\limits_{-0\cdot 5}^{0\cdot 5} k_2(x_i+sh)\,ds,$$

$$F_i = \int\limits_{-0\cdot 5}^{0\cdot 5} k_3(x_i+sh)\,ds, \qquad x_i = ih, \quad i = 1, 2, \ldots, N-1, \quad h = 1/N.$$

This scheme has a number of advantages.

The value of $(L_h^{(k)}u^h)_i$ can be determined using the generating functional

$$\Phi^h[\bar{u},\ \bar{k}(s)] = \frac{1}{h^2}[B^{(\bar{k}_1)}(\bar{u}_1-\bar{u}_0) - A^{(\bar{k}_1)}(\bar{u}_0-\bar{u}_1)] - D^{(\bar{k}_2)}\bar{u}_0 + F^{(\bar{k}_3)},$$

where

$$A^{(\bar{k}_1)} = A[\bar{k}_1(s)] = \left[\int\limits_{-1}^{0}\frac{ds}{\bar{k}_1(s)}\right]^{-1},$$

$$B^{(\bar{k}_1)} = B[\bar{k}_1(s)] = \left[\int\limits_{0}^{1}\frac{ds}{\bar{k}_1(s)}\right]^{-1} = A[\bar{k}_1(1+s)],$$

$$D^{(\bar{k}_2)} = D[\bar{k}_2(s)] = \int_{-0\cdot 5}^{0\cdot 5} \bar{k}_2(s)ds\,, \qquad F^{(\bar{k}_3)} = F[\bar{k}_3(s)] = \int_{-0\cdot 5}^{0\cdot 5} k_3(s)ds\,,$$

defined for the net function $\bar{u} = \{\bar{u}_m\}$ given on the three-point pattern $\mathfrak{M}_0$ ($m = -1, 0, 1$) and functions $\bar{k}(s) = \{\bar{k}_1(s), \bar{k}_2(s), \bar{k}_3(s)\}$, given on the segment $\Sigma_0(-1 \leqslant s \leqslant 1)$, which we shall call the coefficient pattern, if we put

$$\bar{u}_m = u^h_{i+m}\,, \quad m \in \mathfrak{M}_0\,, \quad \bar{k}_j(s) = k_j(x_i + sh), \quad j = 1, 2, 3, \quad s \in \Sigma_0\,.$$

We define the concept of the homogeneous difference scheme $L_h^{(k)}u^h$. We first give some other definitions.

The finite set of points of an integral net, $\mathfrak{M}_0\{m = -m_1, -m_1+1, \ldots \ldots, 0, 1, \ldots, m_2\}$ is said to be a net set, and the segment $\Sigma_0\{-m_1 \leqslant s \leqslant \mathfrak{M}_2)$ will be called a coefficient pattern. The functional $\Phi^h[\bar{u}, \bar{k}(s)]$, defined for the net function $\bar{u}\{\bar{u}_m, m \in \mathfrak{M}_9\}$ and coefficients $\bar{k}(s)$ $(s \in \Sigma_0)$ on the corresponding patterns, and depending on h and a parameter, will be called a generating functional.

In our definition of the net pattern $\mathfrak{M}_0$ and the coefficient pattern Σ_0, we can, without loss of generality, take m_1 and m_2 as identical. In the case of many variables we can make the pattern Σ_0 independent of $\mathfrak{M}_0$, the set of vectors joining the origin of coordinates to the points of the integral net.

The shift transformation $x = \bar{x} + sh$ maps the pattern $\mathfrak{M}_0$ and Σ_0 on the sets of points $\mathfrak{M}^h_{\bar{x}}$ and $\Sigma^h_{\bar{x}}$, which we shall call transformed h-pattern at the point $\bar{x}$.

We consider the functions u^h and $k(x)$ as given, respectively, on the base net S_h and on Σ, the base region of change in x. These functions together with the transformed pattern $\mathfrak{M}^h_{x_i}$ and $\Sigma^h_{x_i}$ at the point x_i induce the functions $\bar{u}^h_{x_i}$ and $\bar{h}^h_{x_i}(s)$ defined on $\mathfrak{M}_0$ and Σ_0:

$$\bar{u}^h_{x_i} = u^h(x_i + mh), \quad m \in \mathfrak{M}_0, \quad \bar{k}^h_{x_i}(s) = k(x_i + sh), \quad s \in \Sigma_0\,,$$

provided that $\mathfrak{M}^h_{x_i}$ and $\Sigma^h_{x_i}$ belong to S_h and Σ respectively.

The difference scheme $L_h^{(k)}u^h$, corresponding to the differential operator $L^{(k)}u$, will be said to be a homogeneous difference scheme in the class of coefficients $k(x) \in K$ if there exists a generating functional $\Phi^h[\bar{u}, \bar{k}(s)]$ such that the values $(L_h^{(k)}u^h)_i$ can be found from the formula:

$$(L_h^{(k)}u^h)_i = \Phi^h[\bar{u}^h_{x_i}, \bar{k}^h_{x_i}(s)] = \Phi^h[u^h(x_i + mh), k(x_i + sh)]\,.$$

For any choice of $k(x) \in K$ the difference scheme $L_h^{(k)}$ defines an operator which transforms the function u^h, given on the net S_h, into the function $L_h^{(k)}u^h$, given on the net $S'_h(x_i, m_1 \leqslant i \leqslant N - m_2)$, which is part of the net S_h.

If the difference scheme $L_h^{(k)}u^h$ is *linear* with respect to the net function $u^h = \{u^h_i\}$, then

$$(L_h^{(k)}u^h)_i = \sum_l a^h_{il}[k(x)]\, u^h_l + b^h_i[k(x)]\,,$$

where the summation is taken, generally speaking, over the whole net S_h. The coefficients $a_{il}^h[k(x)]$ and $b_i^h[k(x)]$ are functionals of the coefficients of the differential operator $L^{(k)}u$, depending on the parameter h. Being given $L_h^{(k)}u^h$ is equivalent to being given the matrix-functional $(a_{il}^h[k(x)])$ and the vector-functional $b_i^h[k(x)]$. If the linear scheme $L_h^{(k)}u^h$ is homogeneous, then

$$(L_h^{(k)}u^h)_i = \sum_{j=-m_1}^{m_2} A_j^h[\bar{k}_i^h(s)]\,u_{i+j}^h + B^h[\bar{k}_i^h(s)]\,, \qquad \left(\bar{k}_i^h(s) = k(x_i+sh)\right),$$

where $A_j^h[\bar{k}(s)]$ and $B^h[\bar{k}(s)]$ are parametric functionals, defined for vector-functions $\bar{k}(s)$ given on the coefficient pattern $\Sigma_0(-m_1 \leqslant s \leqslant m_2)$, where

$$a_{il}^h[k(x)] = A_j^h[k(x_i+sh)]\,, \quad j = l-i\,, \text{ for } -m_1 \leqslant j \leqslant m_2\,,$$

$$a_{il}^h[k(x)] = 0 \qquad \text{for } j < -m_1 \text{ and } j > m_2\,,$$

$$b_i^h[k(x)] = B^h[k(x_i+sh)]\,.$$

The generating functional of this linear homogeneous scheme is equal to

$$\Phi^h[\bar{u}, \bar{k}(s)] = \sum_{j=-m_1}^{m_2} A_j^h[\bar{k}(s)]\bar{u}_j + B^h[\bar{k}(s)]\,,$$

where $\bar{k}(s)$ is a function defined on the pattern $\Sigma_0(-m_1 \leqslant s \leqslant m_2)$.

The difference scheme is said to be *symmetric* if the expression $L_h^{(k)}u^h$ is unchanged when the direction of the x-axis changes. The symmetry condition for a homogeneous scheme is of the form

$$\Phi^h[u_{i+j}, k(x_i+sh)] = \Phi^h[u_{i-j},\ k(x_i-sh)]\,,$$

$$j = 0, \pm 1, \pm 2, \ldots, \pm m \quad (m_1 = m_2 = m)\,, \qquad -m \leqslant s \leqslant m\,.$$

The symmetry of a linear homogeneous scheme is defined by the equalities

$$A_j^h[\bar{k}(s)] = A_{-j}^h[\bar{k}(-s)]\,, \quad j = 0, \pm 1, \ldots, \pm m \quad (m_1 = m_2 = m)\,,$$

$$B^h[\bar{k}(s)] = B^h[\bar{k}(-s)]\,.$$

When solving systems of differential equations $dY/dx = k\left(x, Y(x)\right)$ the class of equations is not simply defined by the coefficients $k(x)$ depending on the variable x, but also by the "coefficients" $k(x, Y)$, depending on the required vector-functions $Y(x)$. The concept of homogeneous difference schemes can be extended to this case also, and includes schemes used in the Euler and Adams–Stermer methods*.

2. *Some examples.* To clarify the questions posed in Section 1, we consider some examples.

Example 1. Consider the first boundary problem

$$L^{(k)}u = \frac{d}{dx}\left[k(x)\frac{du}{dx}\right] = 0\,, \qquad 0 < x < 1\,,\ u(0) = 1\,, \quad u(1) = 0\,.$$

* The Runge–Kutta scheme goes outside this definition of homogeneous schemes. However, it is not difficult to extend the concept of homogeneity of difference schemes to include this scheme too.

Consider the homogeneous difference scheme defined by the generating functional

$$\Phi^h[\bar{u}, \bar{k}(s)] = \bar{k}(0)\frac{\bar{u}_1 - 2\bar{u}_0 + \bar{u}_{-1}}{h^2} - \frac{\bar{k}(1) - \bar{k}(-1)}{2h}\cdot\frac{\bar{u}_1 - \bar{u}_{-1}}{2h},$$

which leads to the system of difference equations

$$(L_h^{(k)}u^h)_i = \Phi^h[u^h(x_i + mh),\ k(x_i + sh)]$$

$$= k_i\frac{u_{i+1}^h - 2u_i^h + u_{i-1}^h}{h^2} + \frac{k_{i+1} - k_{i-1}}{2h}\cdot\frac{u_{i+1}^h - u_{i-1}^h}{2h} = 0,$$

$$m = -1,\ 0,\ 1, \quad 0 < i < N, \quad h = 1/N,$$

or

$$\frac{1}{h^2}[B_i^{(k)}\Delta u_i^h - A_i^{(k)}\nabla u_i^h] = 0, \quad 0 < i < N, \quad u_0^h = 1, \quad u_N^h = 0,$$

where

$$\Delta u_i = u_{i+1} - u_i, \quad \nabla u_i = u_i - u_{i-1}, \quad B_i^{(k)} = k_i + \frac{1}{4}(k_{i+1} - k_{i-1}),$$

$$A_i^{(k)} = k_i - \frac{1}{4}(k_{i+1} - k_{i-1}).$$

As we shall show below (see § 2, Section 4), this scheme gives second order accuracy in the class of smooth coefficients $k(x)$ $(k \in C^{(3)})$

$$u_i^h - u(x_i) = O(h^2).$$

We show that this scheme does not even give convergence in the class of discontinuous coefficients. Consider the piece-wise constant function

$$k(x) = \begin{cases} k_1, & 0 \leqslant x < \xi, \\ k_2, & \xi < x \leqslant 1, \end{cases}$$

where ξ is an irrational number. Let ξ belong to the interval $(x_n,\ x_{n+1})$ of the net S_h, $\xi = x_n + \theta h$, $0 < \theta < 1$. The difference equations for $i \neq n$, $i \neq n+1$ give

$$\Delta u_i^h = \nabla u_{i+1}^h,$$

i.e. u_i^h is linear with respect to the suffix i for $i < n$ and $i > n+1$

$$u_i^h = \begin{cases} 1 - \dfrac{\alpha}{\Delta}x_i, & x_1 < \xi, \\[2ex] \dfrac{\beta}{\Delta}(1 - x_i), & x_i > \xi. \end{cases}$$

The coefficients $\dfrac{\alpha}{\Delta}$ and $\dfrac{\beta}{\Delta}$ are determined from the equations for $i = n, n+1$:

$$\alpha = \frac{3 + \varkappa}{5 - \varkappa}, \quad \beta = \frac{3\varkappa + 1}{5\varkappa - 1}, \quad \varkappa = \frac{k_2}{k_1},$$

$$\Delta = \alpha\xi + \beta(1 - \xi) + h[1 - (\beta - \alpha)(1 - \theta)].$$

Introducing the polygonal function $\tilde{u}^h(x, h)$ stretching along the net function u_i^h and passing to the limit as $h \to 0$ we see that the limit function

$$\tilde{u}(x) = \lim_{h\to 0} \tilde{u}^h(x, h) = \begin{cases} 1 - \dfrac{\alpha}{\Delta_0} x, & x < \xi, \\ \dfrac{\beta}{\Delta_0}(1-x), & x < \xi \end{cases}$$

$$\left(\Delta_0 = \alpha\xi + \beta(1-\xi)\right)$$

is different from the solution of the original problem, which is equal to

$$u(x) = \begin{cases} 1 - \dfrac{\varkappa x}{\varkappa\xi + 1 - \xi}, & x < \xi, \\ \dfrac{1-x}{\varkappa\xi+1-\xi}, & x > \xi, \end{cases}$$

when $\varkappa \neq 1$. Indeed, comparing the expressions for $\tilde{u}(x)$ and $u(x)$ we can show that equality $\tilde{u}(x) = u(x)$ holds only when $\varkappa = 1$. For some $\varkappa$ and ξ we have $\Delta_0 = 0$ and in general the limit function $\tilde{u}(x)$ does not exist.

Thus, when using difference schemes, it is necessary to find the class of coefficients in which they give convergence.

Example 2. Consider the first boundary problem

$$L^{(k,q,f)}u = \frac{d}{dx}\left[k(x)\frac{du}{dx}\right] - q(x)u + f(x) = 0, \quad 0 < x < 1,$$

$$u(0) = \bar{u}_1, \quad u(1) = \bar{u}_2,$$

for the class of differential equations characterised by the coefficients $k, q, f \in Q^{(0)}$ (i.e. by piece-wise continuous coefficients on the segment $0 \leqslant x \leqslant 1$). We shall assume that $k(x) \geqslant M_1 > 0$, $q(x) \geqslant 0$.

It is obvious that in the interval (x_{i-1}, x_{i+1})

$$u(x) = P_i^h(x)u_{i-1} + Q_i^h(x)u_{i+1} + R_i^h(x),$$

where $P_i^h(x)$ and $Q_i^h(x)$ are found in terms of the solution $u^{(1)}(x)$ and $u^{(2)}(x)$ of the homogeneous equation

$$L^{(k,q)}u = \frac{d}{dx}\left[k(x).\frac{du}{dx}\right] - q(x)u = 0,$$

satisfying the conditions

$$u^{(1)}(x_{i-1}) = 0, \quad k_{i-1}u^{(1)\prime}(x_{i-1}) = \frac{1}{h}, \quad u^{(2)}(x_{i+1}) = 0, \quad k_{i+1}u^{(2)\prime}(x_{i+1}) = -\frac{1}{h},*$$

* The normalisation of the derivative is arbitrary. The initial values of the derivatives are chosen for convenience in the subsequent calculations.

where

$$P_i^h(x) = \frac{u^{(2)}(x)}{u^{(2)}(x_{i-1})}, \quad Q_i^h(x) = \frac{u^{(1)}(x)}{u^{(1)}(x_{i+1})}, \quad u^{(1)}(x_{i+1}) \neq 0, \quad u^{(2)}(x_{i-1}) \neq 0$$

$$(q \geqslant 0)$$

and $R_i^h(x)$ is found in terms of the solution $u^{(3)}(x)$ of the initial non-homogeneous equation $L^{(k,q,f)}u = 0$ satisfying the conditions

$$u^{(3)}(x_{i-1}) = u^{(3)}(x_{i+1}) = 0, \quad R^h(x) = u^{(3)}(x).$$

Thus, the exact solution at interpolation points of the net coincides with the solution of the difference equations

$$y_i = P_i^h y_{i-1} + Q_i^h y_{i+1} + R_i^h, \quad y_0 = \bar{u}_1, \quad y_N = \bar{u}_2,$$

whose coefficients

$$P_i^h = P_i^h(x_i), \quad Q_i^h = Q_i^h(x_i), \quad R_i^h = R_i^h(x_i)$$

are functionals of $k(x)$, $q(x)$ and $f(x)$.

We transform the conditions defining P_i^h, Q_i^h and R_i^h. The transformation

$$s = \frac{x - x_i}{h}$$

transforms $u^{(1)}(x)$ and $u^{(2)}(x)$ into the solutions $\bar{u}^{(1)}(s)$ and $\bar{u}^{(2)}(s)$ of the equation

$$\frac{d}{ds}\left(\bar{k}(s)\frac{d\bar{u}}{ds}\right) - h^2\bar{q}(s)\bar{u}(s) = 0, \quad -1 \leqslant s \leqslant 1,$$

$$\bar{k}(s) = k(x_i + sh), \quad \bar{q}(s) = q(x_i + sh),$$

satisfying the conditions

$$\bar{u}^{(1)}(-1) = 0, \quad \bar{k}(-1)\bar{u}^{(1)\prime}(-1) = 1, \quad \bar{u}^{(2)}(1) = 0, \quad \bar{k}(1)\bar{u}^{(2)\prime}(1) = -1,$$

and $u^{(3)}(x)$ into the solution $\bar{u}^{(3)}(s)$ of the non-homogeneous equation

$$\frac{d}{ds}\left(\bar{k}(s)\frac{d\bar{u}^{(3)}}{ds}\right) - h^2\bar{q}(s)\bar{u}^{(3)}(s) + h^2\bar{f}(s) = 0, \quad \bar{f}(s) = f(x_i + sh),$$

defined by the conditions

$$\bar{u}^{(3)}(-1) = \bar{u}^{(3)}(1) = 0.$$

Consider the functionals

$$P^h[\bar{k}(s), \bar{q}(s)] = \frac{\bar{u}^{(2)}(0)}{\bar{u}^{(2)}(-1)}, \quad Q^h[\bar{k}(s), \bar{q}(s)] = \frac{\bar{u}^{(1)}(0)}{\bar{u}^{(1)}(1)},$$

$$R^h[\bar{k}(s), \bar{q}(s), \bar{f}(s)] = \bar{u}^{(3)}(0),$$

depending on $\bar{k}(s)$, $\bar{q}(s)$, $\bar{f}(s)$.

The exact difference scheme is a homogeneous scheme and is defined by the characteristic functional

$$\Phi^h = \bar{u}_0 - P^h \bar{u}_{-1} - Q^h \bar{u}_1 - R^h .$$

However, in practice, the exact scheme is not used due to the complexity of the determination of its coefficients. To solve this boundary problem we make use of various families of "admissible" schemes, whose coefficients are calculated quite simply.

In particular, discrete schemes, defined by the values of the coefficients of the equation at the interpolation points of the net only, are widely used (see, for example, the scheme of example 1).

To construct the theory of homogeneous difference schemes we must choose an initial family of schemes, find a functional class of coefficients of a differential equation in which the schemes from the original family converge, and also discuss the accuracy of the separate schemes.

3. *The approximation and accuracy of difference schemes* [6]. We consider the class of differential equations $L^{(k)}u = 0$ defined for $k(x) \in K$, and the homogeneous difference scheme $L_h^{(k)}u^h$ defined for the same class of coefficients.

Let

$$\Phi^h[\bar{u}, \bar{k}(s)]$$

be the generating functional of the scheme $L_h^{(k)}u^h$, defined on the patterns $\mathfrak{M}_0(-m_1 \leqslant m \leqslant m_2)$ and $\Sigma_0(-m_1 \leqslant s \leqslant m_2)$. Let $\bar{x}$ be a fixed point in the region of definition of the operator $L^{(k)}v$ and let $v(x)$ be some function given in a neighbourhood of this point. With the pattern

$$\mathfrak{M}_{\bar{x}}^h = (\bar{x} + mh, \ m \in \mathfrak{M}_0)$$

the function $v(x)$ induces the net function

$$\bar{v}_{\bar{x}}^h = \{v(\bar{x} + mh), \ m \in \mathfrak{M}_0\}$$

on the pattern $\mathfrak{M}_0$.

Consider the quantity

$$L_h^{(k)}\bar{v}_{\bar{x}}^h = \Phi^h[v(\bar{x} + mh), \ k(\bar{x} + sh)]$$

and the difference

$$\varphi(\bar{x}, v, h) = L_h^{(k_0)}\bar{v}_{\bar{x}}^h - (L^{(k_0)}v)_{x=\bar{x}},$$

where $k_0(x)$ is some fixed coefficient of the class K; we call the function φ the approximation error at the point $\bar{x}$ due to approximating to the operator $L_h^{(k_0)}$ by the operator $L^{(k_0)}$, or the approximation error of the operator $L_h^{(k_0)}$.

We shall say that the difference operator $L_h^{(k_0)}$ has an approximation of the nth order as $h \to 0$ at the point $\bar{x}$ with respect to the differential operator $L^{(k_0)}$ if we can find m such that for any function $v(x) \in C^{(m)}$, which is differentiable m times.

$$\varphi(\bar{x}, v, h) = O(h^n) \quad \text{or} \quad |\varphi(\bar{x}, v, h)| \leqslant Mh^n,$$

where M is a constant depending on $\bar{x}$, $v(x)$ and $k_0(x)$.

We shall say that the difference scheme $L_h^{(k)}$ at the point $\bar{x}$ is of nth order approximation to $L^{(k)}$ in the class K of coefficients if for any fixed coefficient $k_0(x) \in K$ the difference operator $L_h^{(k_0)}$ is of nth order approximation at the point $\bar{x}$.

Let $u(x)$ be a solution of the equation

$$L^{(k)}u = 0,$$

satisfying some supplementary conditions $lu = 0$ (problem I), and let $u^h = \{u_i^h\}$ be a solution of the difference equation

$$L_h^{(k)}u = 0$$

with corresponding supplementary conditions $l_h u^h = 0$ (problem II).

The difference $z_i^h = u_i^h - u(x_i)$ is the error in the solution of the initial problem; in future we shall evaluate this difference according to the norm $\| z_i^h \|_1 = \max\limits_i |z_i^h|$ where the maximum is taken over all the points of the net on which the function z_i^h is given.

We shall say that the solution of the difference problem II converges to that of problem I in the class K if

$$\| z_i^h \|_1 \leqslant \rho(h) \to 0 \quad \text{as} \quad h \to 0$$

for any coefficient $k(x) \in K$.

Further, if $\| z_i^n \|_1 = O(h^n)$ or $\| z_i^n \|_1 \leqslant Mh^n$, where M is a constant depending only on the choice of the coefficient $k(x)$, then we shall say that problem II is of nth order accuracy.

The accuracy of the difference problem depends both on the choice of the difference scheme and on the choice of the difference boundary conditions. If the boundary conditions for problems I and II are the same (as, for example, in the first boundary problem, in the introduction, section 4) then the accuracy of the difference boundary problem is completely determined by the choice of difference scheme. In this case we can say that "the difference scheme converges", or "the difference scheme has nth order accuracy".

If the scheme $L_h^{(k)}$ is linear, then the equation for the function z_i^h is

$$L_h^{(k)} z_i^h = -\varphi_i^h, \quad \text{where} \quad \varphi_i^h = L_h^{(k)} u_i - (L^{(k)}u)_i$$

is the approximation error at the point x_i for the difference operator $L_h^{(k)}$ taken over the solution $u(x)$ of the differential equation $L^{(k)}u = 0$.

The orders of approximation considered over the family of sufficiently smooth functions $v(x) \in C^{(m)}$ and over the family of solutions of the equations $L^{(k)}u = 0$ may differ essentially.

We shall say that the homogeneous difference schemes $\bar{L}_h^{(k)}$ and $\bar{\bar{L}}_h^{(k)}$ defined in the same class K are nth order equivalent as regards approximation at the point $x = \bar{x}$ if, for any $k(x) \in K$

$$\varphi(\bar{x}, v; h) = (\bar{L}_h^{(k)} v)_{x=\bar{x}} - (\bar{\bar{L}}_h^{(k)} v)_{x=\bar{x}} = O(h^n),$$

where $v(x)$ is any sufficiently smooth function, i.e. $|\Phi(\bar{x}, v; h)| \leqslant Mh^n$, where M is a constant depending on the choice of $k(x)$, $v(x)$ and $\bar{x}$.

Let $\bar{y}_i^h$ and $\bar{\bar{y}}_i^h$ be solutions of the equations $\bar{L}_h^{(k)} \bar{y}_i^h = 0$ and $\bar{\bar{L}}_h^{(k)} \bar{\bar{y}}_i^h = 0$ with the supplementary conditions $\bar{l}_h^{(k)} \bar{y}^h = 0$ and $\bar{\bar{l}}_h^{(k)} \bar{\bar{y}}^h = 0$ (problems II′ and II″).

We shall say that the difference boundary problems II′ and II″ are nth order equivalent in the class K if for any $k(x) \in K$

$$\|\bar{y}_i^h - y_i^h\|_1 < Mh^n,$$

where M is a constant depending on the choice of $k(x)$ and independent of h.

It is clear that we can always replace difference problems by their equivalents and select the structurally simplest difference schemes from the class of equivalent problems.

4. *Basic results.* In this article we study homogeneous schemes for the solution of the first boundary problem

$$L^{(k, q, f)} u = \frac{d}{dx}\left[k(x)\frac{du}{dx}\right] - q(x) u + f(x) = 0 \qquad (0 < x < 1),$$

$$u(0) = \bar{u}_1, \quad u(1) = \bar{u}_2, \tag{1}$$

whose coefficients k, q, f are piece-wise continuous functions $k, q, f \in Q^{(0)}$, where $k(x) \geqslant M > 0$, $q(x) \geqslant 0$.

In § 1 we take as our initial family of difference schemes the three-point homogeneous difference schemes $L_h^{(k, q, f)}$ characterised by the linear generating functional

$$\Phi^h[\bar{u}(m), \bar{k}(s), \bar{q}(s), \bar{f}(s)] = \frac{1}{h^2}[B^{(h, \bar{k})}(\bar{u}_1 - \bar{u}_0) - A^{(h, \bar{k})}(\bar{u}_0 - \bar{u}_{-1})] - D^{(h, \bar{q})}\bar{u}_0 + F^{(h, \bar{f})},$$

each coefficients of which is a functional of only one coefficient of the differential equation (I):

$$\left.\begin{aligned} A^{(h, k)} &= A^h[\bar{k}(s)], & B^{(h, \bar{k})} &= B^h[\bar{k}(s)], \\ D^{(h, \bar{q})} &= D^h[\bar{q}(s)], & F^{(h, \bar{f})} &= F^h[\bar{f}(s)] \end{aligned}\right\}, \qquad -1 \leqslant s \leqslant 1.$$

Here D^h and F^h are linear functionals. Such schemes are commonly used in practice, and we call them standard schemes.

The value of a scheme can be characterised by its approximation error

$$\varphi(\bar{x}, u; h) = (L_h^{(k, q, f)} u)_{x=\bar{x}} - (L^{(k, q, f)} u)_{x=\bar{x}},$$

where $u(x)$ is a solution of equation (I).

To find the order of approximation as $h \to 0$ the function $\varphi(\bar{x}, u; h)$ must be expanded in the parameter h and the coefficients of powers of h up to the rth must be calculated. It is possible to do this if the pattern functionals A^h, B^h, D^h and F^h have derivatives of the corresponding order both with respect to the parameter h and with respect to their functional argument.

We define the rank of a functional, including in our definition the requirements of homogeneity, monotonicity and normalisation as well as differentiability.

Using the concept of the rank of template functionals, we consider the different classes $\mathfrak{L}(n_1, n_2, n_3)$ of schemes whose functionals A^h and B^h have rank n_1, and D^h and F^h have rank n_2, n_3 and are defined on the segment $-0{\cdot}5 \leqslant s \leqslant 0{\cdot}5$.

If $n_1 = n_2 = n_3 = n$ then we shall say that $L_h^{(k, q, f)}$ is a scheme of the nth rank.

We consider special families of schemes: those which are conservative, or self-conjugate ($B^h[\bar{k}(s)] = A^h[\bar{k}(1+s)]$), those which are discrete and those which are canonical, their pattern functionals not dependent on the parameter h.

Having found necessary and sufficient conditions for the scheme $L_h^{(k, q, f)}$ $(n = 1, 2)$ to have nth order approximation, in the form of a number of relations (A.C.n) between the moments of the functionals of the scheme, we pass in § 2 to a study of the questions of convergence and accuracy of the original schemes in some class of mooth coefficients $C^{(m)}$*. It is proved that in order that the original scheme $L_h^{(k, q, f)}$ of the class $\mathfrak{L}(n+1, n, n)$ with coefficients $k(x) \in C^{(m_k)}$, $m_k \geqslant n+1$, $q(x) \in C^{(m_q)}$, $m_q \geqslant n$; $f(x) \in C^{(m_f)}$, $m_f \geqslant n$ shall be accurate to the nth order, it is both necessary and sufficient that it has nth order approximation (Theorem 1).

To prove this theorem we make use of Green's difference function of the operator $L_h^{(k, q)}$. In Section 2 we give the construction of Green's function, and in Section 3 we give uniform upper and lower bounds for Green's function and also for its first difference ratios.

We note that in studying the convergence and accuracy of schemes in the class of smooth coefficients, we use the norm

$$\| \psi \|_1 = \max_{0 < i < N} |\psi_i| ,$$

and in the class of discontinuous coefficients, of the norms

$$\| \psi \|_3 = \sum_{i=1}^{N-1} |\psi_i| h \quad \text{and} \quad \| \psi \|_2 = \sum_{i=1}^{N-1} h \left| \sum_{s=1}^{i} \psi_s h \right| .$$

Although in the class $C^{(m)}$ the order of accuracy of the scheme $L_h^{(k, q, f)}$ is the same as the order of approximation, there is no such connection in the class of discontinuous coefficients. It is sufficient to recall Example 1 given above. The scheme there has second order approximation in the class $C^{(m)}$ $(m \geqslant 3)$ (this can easily be verified) and yet diverges in the class of discontinuous coefficients $k(x) \in Q^{(m)}$ (for any $m \geqslant 0$).

* See Section 1, § 1 for the definitions of C^m and Q^m.

We find, by studying the function $\varphi(\bar{x}, u; h)$ at points of the net adjacent to a point of discontinuity ξ $(x_n \leqslant \xi \leqslant x_{n+1})$ of the coefficient $k(x)$, that the approximation error for φ_n^h and φ_{n+1}^h when $x = x_n$ and $x = x_{n+1}$ in general tends to infinity as $h \to 0$. However, there is still a possibility that the solution of the difference equation converges to the solution of equation (I). We can then ask what properties the scheme $L_h^{(k,q,f)}$ must possess for there to convergence in the class $Q^{(m)}$.

In § 3 we prove a basic lemma giving the necessary condition which the scheme $L_h^{(k,q,f)}$ must satisfy for convergence in the class $Q^{(m)}$. This condition has the form

$$\Delta(\xi, h) = h(B_n^h \varphi_{n+1}^h + A_{n+1}^h \varphi_n^h) = \rho(h) \to 0 \tag{b}$$

or

$$\frac{B_n^h B_{n+1}^h}{k_+} - \frac{A_n^h A_{n-1}^h}{k_-} = \rho(h) \to 0 \quad \text{as} \quad h \to 0, \tag{b'}$$

where $k_- = k(\xi - 0)$, $k_+ = k(\xi + 0)$.

If we require the scheme $L_h^{(k,q,f)}$ in $Q^{(m)}$ to have 2nd order accuracy, then the two conditions:

$$h^2 \varphi_n^h = O(h^2), \qquad h^2 \varphi_{n+1}^h = O(h^2), \tag{a$_2$}$$

$$\Delta(\xi, h) = O(h^2). \tag{b$_2$}$$

must be satisfied.

It should be noted that any conservative scheme of zero rank satisfies the necessary condition for convergence.

It is proved that condition (b) is not only necessary for a scheme of type $\mathcal{L}(1, 0, 0)$ but also sufficient for convergence of the scheme $L_h^{(k,q,f)}$ in the class of coefficients $k(x) \in Q^{(1)}$, $q, f \in Q^{(0)}$ (Theorem 3, § 3).

A certain error is always admissible, generally speaking, in calculating the coefficients of the difference scheme. This can occur because of insufficient information about the coefficients k, q, f of equation (I): for example, the functions $k(x)$, $q(x)$ and $f(x)$ may have to be determined approximately (using some computing algorithm) on a discrete set of points. Moreover, it can happen that the pattern functionals of the scheme are only approximate.

It is therefore clear why the problem of schemes with disturbed coefficients is so important.

In § 4 we introduce the norm of the disturbance of coefficients of a scheme, and using it we give a definition of coefficient-stable (co-stable) difference schemes. For a small distortion in the coefficients of the scheme the "disturbed" scheme must converge as $h \to 0$ in $Q^{(m)}$, i.e. $\|\tilde{y} - u\|_1 = \rho(h) \to 0$ as $h \to 0$ if

$$\|\tilde{A}^h - A^h\|_3 = \sum_{i=1}^{N-1} |\tilde{A}_i^h - A_i^h| h = \rho(h), \qquad \|\tilde{B}^h - B^h\|_3 = \rho(h);$$

$$\|\tilde{D}^h - D^h\|_3 = \rho(h), \qquad \|\tilde{F}^h - F^h\|_3 = \rho(h),$$

(all the $\rho(h) \to 0$ as $h \to 0$) where $\tilde{y}_i$ is a solution of the difference boundary problem with disturbed coefficients $\tilde{A}_i^h$, $\tilde{B}_i^h$, $\tilde{D}_i^h$, $\tilde{F}_i^h$ and $u(x)$ is a solution of problem I.

It is proved that the necessary and sufficient condition for the coefficient-stability of a canonical scheme is that it shall be conservative. In the subsequent sections we only consider conservative schemes.

In § 5 we study questions of the convergence and accuracy of conservative difference schemes. It is proved that a conservative scheme of zero rank converges in the class of piece-wise continuous coefficients ($k, q, f \in Q^{(0)}$); any conservative scheme of the first rank has first order accuracy in the class of coefficients $Q^{(m)}$ $(m \geqslant 1)$; any conservative scheme $L_h^{(k,q,f)}$ of the second rank, satisfying the conditions for second order approximation (A.C.2) has second order accuracy in the class $C^{(2)}$, and, generally speaking, first order accuracy in the class $Q^{(m)}$ (for any $m \geqslant 1$).

It should be noted that in proving these theorems of the convergence and accuracy of difference schemes we make use of the a priori estimate established in § 2:

$$\|z\|_1 \leqslant M \|\varphi\|_2,$$

where z is the error of the solution of the difference boundary problem, and φ is the approximation error of the scheme $L_h^{(k,q,f)}$ over the solution of problem (I).

The estimate of the approximation error φ according to the norm $\| \ \|_2$ allows us, moreover, to lower the rank of the pattern functionals and the order m of the classes $C^{(m)}$ or $Q^{(m)}$ of coefficients of equation (I).

Thus, the results we have listed answer the question we posed above concerning "open" computing schemes, suitable for solving boundary problems for equation (I) in the class $Q^{(m)}$ $(m \geqslant 0)$ of discontinuous coefficients without explicitly finding the points of discontinuity. As we show open computing schemes belong to the family of conservative difference schemes.

The results we obtain are used to construct open computing difference schemes for solving parabolic type equations with discontinuous coefficients (see [20]).

In conclusion, we mention some questions which go beyond the present article. Our results can be applied without essential change to the class of boundary problems corresponding to the boundary conditions of the third kind. We shall not consider here the very important questions regarding schemes of second order accuracy in the class of discontinuous coefficients, the best canonical scheme of second order accuracy, the accuracy of difference methods for solving the Sturm–Liouville problem in the class of discontinuous coefficients, or homogeneous difference schemes on non-uniform nets.

Our results also pose a number of similar questions for the case of many unknowns. These questions will be considered in later articles.

§ 1. THE INITIAL FAMILY OF DIFFERENCE SCHEMES

In § 1 we discuss the characteristics of families of difference schemes for the differential equation (I) in the class $Q^{(0)}$ of piece-wise continuous coefficients. We consider difference schemes from this family, including canonical, discrete and conservative schemes which are of value for the subsequent theory.

1. *The boundary problem for the differential equation.* To amplify the questions posed in the Introduction about homogeneous difference schemes we take the example of an ordinary differential equation

$$L^{(k,q,f)}u = L^{(k)}u - q(x)u + f(x) = 0, \quad 0 < x < 1, \tag{1}$$

where $L^{(k)}u = \frac{d}{dx}\left[k(x)\frac{du}{dx}\right]$.

We shall consider the first boundary problem

$$L^{(k,q,f)}u = 0, \quad 0 < x < 1, \quad u(0) = \bar{u}_1, \quad u(1) = \bar{u}_2(1). \tag{I}$$

The class of boundary problems (I) is defined if we can find the families of functions to which the coefficients $k(x)$, $q(x)$ and $f(x)$ belong.

Let $C^{(n)}[a, b]$ be the class of functions with a continuous derivative on the segment $a \leqslant x \leqslant b$, and let $Q^{(n)}[a, b]$ be the class of functions which are piece-wise continuous on the segment $[a, b]$ with piece-wise continuous derivatives up to the n order inclusive on this segment. We shall denote the class of functions $C^{(n)}[0, 1]$, $(Q^n[0, 1])$ by $C^{(n)}$ $(Q^{(n)})$.

We shall always assume that the coefficients of equation (I) satisfy the conditions

$$0 < M_1 \leqslant k(x) \leqslant M_2, \quad 0 \leqslant q(x) \leqslant M_3, \quad |f(x)| \leqslant M_4, \tag{α}$$

where M_j $(j = 1, 2, 3, 4)$ are positive constants. We include the conditions (α) in our statement of problem (I).

We require the solution of problem (I) to have the following properties: (1) if $k(x) \in C^{(r+1)}$, $q(x) \in C^{(r)}$, $f(x) \in C^{(r)}$ then $u(x) \in C^{(r+2)}$ $(r \geqslant 0)$; (2) if $k(x) \in Q^{(m)}$ $(m \geqslant 0)$ and there is a discontinuity at the point $x = \xi$ $(k_- \neq k_+$ where $k_- = k(\xi - 0)$, $k_+ = k(\xi + 0))$ then the solution of equation (1) satisfies the conjugacy conditions $u(\xi - 0) = u(\xi + 0) = u(\xi)$, $k_- u'_- = k_+ u'_+$ or $[u] = 0$, $[ku'] = 0$ when $x = \xi$.

2. *The initial family of homogeneous difference schemes.* Consider, on $0 \leqslant x \leqslant 1$, the net $S_h = \{x_0 = 0, \ x_1 = h, \dots, x_i = ih, \ \dots, \ x_N = Nh = 1\}$ and let y_i be a net function.

We set the boundary problem (I) in correspondence with the difference boundary problem

$$L_h^{(k,q,f)}y_i = 0, \quad 0 < i < N, \quad y_0 = \bar{u}_1, \quad y_N = \bar{u}_2, \tag{II}$$

where $L_h^{(k,q,f)}$ is a homogeneous standard three-point difference scheme defined by the formulae

$$L_h^{(k,q,f)}y_i = L_h^{(k)}y_i - D_i^{(h,q)}y_i + F^{(h,f)}, \tag{2}$$

$$L_h^{(k)}y_i = \frac{1}{h^2}\left[B_i^{(h,k)}\Delta y_i - A_i^{(h,k)}\nabla y_i\right] \quad (\Delta y_i = y_{i+1} - y_i, \quad \nabla y_i = y_i - y_{i-1}). \tag{3}$$

The schemes we are considering are peculliar in that each of the coefficient of the difference scheme $L_h^{(k,q,f)}$ is a functional of only one coefficient of the differential equation (1). We assume that the standard three-point difference scheme is homogeneous. This means that it has a generating function of the form

$$\Phi^h[\bar{v}(m), \bar{k}(s), \bar{q}(s), \bar{f}(s)] = \frac{1}{h^2}[B^{(\bar{h},k)}\Delta \boldsymbol{v}_0 - A^{(\bar{h},k)}\nabla\bar{v}_0] - D^{(h,\bar{q})}\bar{v}_0 + F^{(h,\bar{f})}, \quad (4)$$

where

$$m = -1, 0, 1,$$
$$A^{(h,\bar{k})} = A^h[\bar{k}(s)], \qquad B^{(h,\bar{k})} = B^h[\bar{k}(s)], \quad (5)$$
$$D^{(h,\bar{q})} = D^h[\bar{q}(s)], \qquad F^{(h,\bar{f})} = F^h[\bar{f}(s)],$$

the pattern functionals $A^h[\psi(s)]$, $B^h[\psi(s)]$, $D^h[\psi(s)]$, $F^h[\psi(s)]$ being defined for piece-wise continuous functions $\psi(s) \in Q^{(0)}[-1,1]$ given for $-1 \leqslant s \leqslant 1$ and depending, generally speaking, on the parameter h.

The coefficients of the difference scheme are calculated at each point according to the formulae

$$A_i^{(h,k)} = A^h[\bar{k}_i(s)], \quad B_i^{(h,k)} = B^h[\bar{k}_i(s)], \quad k_i(s) = k(x_i + sh), \quad (6)$$
$$D_i^{(h,q)} = D^h[\bar{q}_i(s)], \quad \bar{q}_i(s) = q(x_i + sh); \quad F_i^{(h,f)} = F^h[\bar{f}_i(s)], \quad \bar{f}_i(s) = f(x_i + sh)$$

where $f_i(s) = f(x_i + sh)$.

From the definition of a homogeneous scheme we have

$$L_h^{(k,q,f)} v_i = \Phi^h[v(x_i + mh), k(x_i + sh), q(x_i + sh), f(x_i + sh)], \quad (7)$$

where $m = -1, 0, 1$, $-1 \leqslant s \leqslant 1$.

It should be noted that the fact that the considered difference scheme is three-point is a consequence of its homogeneity.

The difference boundary problem reduces to the solution of a system of linear algebraic equations in the unknowns $y_1, y_2, \ldots, y_{N-1}$. One can easily verify (see § 2) that this problem is always soluble if the conditions

$$A_i^{(h,k)} > 0, \quad B_i^{(h,k)} > 0, \quad D_i^{(h,q)} \geqslant 0,$$

hold at all point $i = 1, 2, \ldots, N-1$.

When we use the term "the initial family of difference schemes" we shall mean homogeneous three-point standard schemes defined by formulae (2), (3) and (6).

3. *Functionals of the* rth *rank*. For more detailed characteristics of the initial family of difference schemes we must know the class of its pattern functionals A^h, B^h, D^h and F^h. We shall need to expand the coefficients of the scheme (for example $A_i^{(h,k)} = A^h[k(x_i + sh)]$) in powers of h; it is possible to do this if the pattern functionals themselves possess some properties of differentiability both with respect to their functional argument and with respect to the parameter h.

$A^h[\psi]$ is called a functional of the rth rank ($r \geqslant 0$) if the following conditions hold:

I.$^{(r)}$ There is an expansion

$$A^h[\psi] = \sum_{\sigma=0}^{r} h^\sigma A^{(\sigma)}[\psi] + h^r \rho(h, \psi), \tag{8}$$

where $|\rho(h, \psi)| \leqslant \rho(h)$*, if $|\psi| \leqslant M$ (M is a positive constant). Each of the functionals $A^{(\sigma)}[\psi]$ ($\sigma = 0, 1, 2, \ldots, r$) has a differential**, of order $r-\sigma$ for any function $\psi \in Q^{(0)}$; this means, for example, that for $A^{(0)}[\psi]$ we can write

$$A^{(0)}[\psi + \delta . \varphi] = A^{(0)}[\psi] + \delta . A_1^{(0)}[\psi, \varphi] + \ldots + \delta^r A_r^{(0)}[\psi, \varphi] + \delta^r \rho(\delta, \psi, \varphi), \tag{9}$$

where $|\rho| \leqslant \rho(\delta)$ if $|\varphi| \leqslant M$ ($\rho(\delta) \to 0$ as $\delta \to 0$).

II.$^{(r)}$ The functional $A^h[\psi]$ and, therefore, all the functionals $A^{(\sigma)}[\psi]$ ($\sigma = 0, 1, \ldots, r$) are homogeneous functionals of the first degree:

$$A^h[c\psi] = cA^h[\psi], \quad A^{(\sigma)}[c\psi] = cA^{(\sigma)}[\psi], \tag{10}$$

where c is a positive constant.

III.$^{(r)}$ The functionals $A^h[\psi]$ and $A^{(\sigma)}[\psi]$ ($\sigma = 0, 1, \ldots, r$) are non-decreasing, i.e.

$$A^h[\psi_2] \geqslant A^h[\psi_1], \quad \text{if} \quad \psi_2 \geqslant \psi_1, \tag{11}$$

where $A^h[\psi]$ is a normalised functional, i.e.

$$A^h[1] = 1 . \tag{12}$$

If $A^h[\psi]$ is a linear functional, then all the functionals $A^{(\sigma)}[\psi]$ are also linear. Therefore the differentiability requirement in condition I$^{(r)}$ is automatically satisfied. Moreover, condition II$^{(r)}$ is an immediate consequence of the linearity.

If $A^h[\psi]$ is a functional of the rth rank, then

$$A^{(0)}[1] = 1, \quad A^{(\sigma)}[1] = 0 \quad \text{for} \quad \sigma = 1, 2, \ldots, r. \tag{13}$$

We shall use the notation

$$A_m^{(\sigma)}[\varphi] = A_m^{(\sigma)}[1, \varphi] \quad (\psi \equiv 1). \tag{14}$$

Let $A^h[\psi]$ be a functional of the rth rank. From I$^{(r)}$ and II$^{(r)}$ we know that for any $k(x) \in C^{(r)}$ satisfying the condition $k(x) \geqslant M_1 > 0$ there is an expansion

$$A_i^{(h,k)} = A^h[k(x_i + sh)] = k_i + hk_i' A_1^{(0)}[s] + h^2 \left\{ k_i' A_1^{(1)}[s] + \frac{(k_i')^2}{k_i} A_2^{(0)}[s] + \right. \\ \left. + \frac{k_i''}{2} A_1^{(0)}[s^2] \right\} + \ldots + h^r \rho(h) . \tag{15}$$

* $\rho(h)$ will in future be used to denote quantities which tend to zero as $h \to 0$. We can use the same symbol in the case of several arguments to denote an expression which tends uniformly to zero as $h \to 0$.

** Cf. M. A. Lavrent'ev and L. A. Lyusternik, "Osnovy variatsionnogo ischisleniya" (The Essentials of Variational Calculus), Vol. I, part II. M.-L. (1935), Ch. VI.

For from $\mathrm{II}^{(r)}$

$$A_i^{(h,k)} = A^h\left[k_i\left(1+\frac{k(x_i+sh)-k_i}{k_i}\right)\right]$$

$$= k_i A^h\left[1+\frac{h}{k_i}\left(k_i's+\frac{k_i''s^2}{2}h+\ldots+h^{r-1}\rho(h)\right)\right]. \qquad (16)$$

Now using the expansions of $\mathrm{I}^{(r)}$ and taking into account the linear property of $A_1^{(0)}[\psi]$, the quadratic property of $A_2^{(0)}[\psi]$, and so on we arrive at (15).

We note some properties of homogeneous functionals of the first degree, omitting proofs.

LEMMA 1. *Any homogeneous functional $A[\psi]$ of the first degree having a differential of the first order $A_1[f, \varphi]$, can be put in the form*

$$A[f] = A_1[f, f].$$

This lemma is the analogue of Euler's well known theorem on homogeneous functions.

LEMMA 2. *If $A[f]$ is a homogeneous functional of the first degree, and $A_k[f, \varphi]$ is its* kth *order differential, then*

$$A_k[cf, \varphi] = \frac{1}{c^{k-1}} A_k[f, \varphi], \qquad k = 1, 2, \ldots, \qquad (17)$$

where c it any positive constant. In particular, $A_1[cf, \varphi]$ is independent of c:

$$A_1[cf, \varphi] = A_1[f, \varphi].$$

LEMMA 3. *The differential $A_1[f, \varphi]$ of a non-decreasing normalised homogeneous functional of the first degree, $A[f]$, is a linear positive functional in the argument φ.*

LEMMA 4. *If $A[f]$ is a non-decreasing normalised homogeneous functional of the first degree, having a first order differential, then*

$$0 \leqslant \frac{A_1[f, \varphi]}{A[f]} \leqslant 1, \quad \textit{if} \quad 0 \leqslant \varphi \leqslant f, \quad f \geqslant \varepsilon > 0. \qquad (18)$$

These lemmas will be used below (for example in § 4, § 6 and elsewhere).

4. *The classes $\mathfrak{Q}(n_1, n_2, n_3)$ of difference schemes.* Consider the difference boundary problem (II) and compare its solution y_i^h with the solution $u(x)$ of problem (I). The accuracy of the solution of problem (II) is characterised by the difference

$$z_i^h = y_i^h - u(x_i), \qquad (19)$$

which, as can easily be observed, is defined by the conditions

$$L_h^{(k,q)} z_i^h = -\varphi_i^h, \quad 0 < i < N, \quad z_0^h = 0, \quad z_N^h = 0, \qquad (\mathrm{III})$$

where

$$L_h^{(k,q)} z_i^h = L_h^{(k)} z_i^h - D_i^{(h,q)} z_i^h, \qquad (20)$$

and

$$\varphi_i^h = L_h^{(k,q,f)}u_i - (L^{(k,q,f)}u)_i \tag{21}$$

denotes the approximation error of the scheme $L_h^{(k,q,f)}$ at the nodal point $x = x_i$ of the net S_h taken over the solution $u(x)$ of problem (I).

Due to the homogeneity of the scheme, the approximation error can be considered at any fixed point $x = \bar{x}$ of the interval $0 < x < 1$:

$$\varphi(x, v; h) = (L_h^{(k,q,f)}v)_{x=\bar{x}} - (L^{(k,q,f)}v)_{x=\bar{x}},$$

where $v(x)$ is any sufficiently smooth function, $h \leqslant h_0 < 1$, h_0 is the distance between the point $x = \bar{x}$ and the boundary of the segment [0, 1].

In order to go into the question of the order of approximation of the scheme we must have an expansion of the function $\varphi(x, v; h)$ in a series of powers of h. To do this we must formulate the differentiability properties of pattern functionals, using the definition of the rank of a functional.

We shall say that the difference scheme $L_h^{(k,q,f)}$ from the initial family belongs to the class of schemes $\mathfrak{L}(n_1, n_2, n_3)$ if the functionals $B^h[\psi]$ and $A^h[\psi]$ are of rank n_1, and the functionals $D^h[\psi(s)]$ and $F^h[\psi(s)]$ are of rank n_2 and rank n_3 respectively, and are linear and defined for functions of $Q^{(0)}$ given on the segment $-0{\cdot}5 \leqslant s \leqslant 0{\cdot}5$, $(\psi \in Q^{(0)}[-0{\cdot}5,\ 0{\cdot}5])$.

We define similarly the classes $\mathfrak{L}(n_1)$ and $\mathfrak{L}(n_1, n_2)$ for schemes $L_h^{(k)}$ and $L_h^{(k,q)}$ respectively.

The schemes $L_h^{(k,q,f)}$ of the class $\mathfrak{L}(n_1, n_2, n_3)$ are called schemes of the nth rank (all the pattern functions are of rank n). Any scheme $L_h^{(k)}$ of $\mathfrak{L}(n)$ is a scheme of the nth rank.

5. *The error of approximation.* We now proceed to the calculation of the approximation error for schemes from the class $\mathfrak{L}(n_1, n_2, n_3)$ (usually $n_2 = n_3$).

We consider the scheme $L_h^{(k,q,f)}$ of the class $\mathfrak{L}(n+1, n, n)$ and assume that $k(x) \in C^{(n+1)}$, $q(x) \in C^{(n)}$, $f(x) \in C^{(n)}$ and the function $v(x) \in C^{(n+2)}$ where $n \geqslant 0$; the solution $u(x)$ of equation (1), in particular, belongs to $C^{(n+2)}$, if k, q, f satisfy the requirements given above in Section 1; therefore we can take $v = u(x)$.

We use the expansions of $A_i^{(h,k)}$ and $B_i^{(h,k)}$ given by formula (15) for $r = n+1$, and for $D_i^{(h,q)}$ (and $F_i^{(h,f)}$) we use the expansion in powers of h:

$$D_i^{(h,q)} = D^h[q(x_i+sh)] = \sum_{\sigma=0}^{n} D_s^{(\sigma)}[q(x_i+sh)]\, h^\sigma + h^n\rho(h)$$

$$= q_i + hq_i' D^{(0)}[s] + \ldots + h^n\rho(h). \tag{22}$$

Then, using the expansion of the function $v(x)$ in the neighbourhood of the point $x = \bar{x}$

$$v(\bar{x}+sh) = v(\bar{x}) + shv'(\bar{x}) + \ldots + \frac{(sh)^{n+2}}{(n+2)!}v^{(n+2)}(\bar{x}) + h^{n+2}\rho(h), \tag{23}$$

we obtain

$$\varphi(x, v; h)|_{x=\bar{x}} = \varphi^{(0)} + h\varphi^{(1)} + \ldots + h^n\varphi^{(n)} + h^n\rho(h).$$

The coefficients $\varphi^{(j)} = \varphi^{(j)}(x,v)|_{x=\bar{x}}$ $(j = 0, 1, —n)$ of the powers h^j depend on the functions v, k, q, f and their derivatives. We shall need expressions for $\varphi^{(0)}$ and $\varphi^{(1)}$:

$$\varphi^{(0)} = a_0 k'(\bar{x})v'(\bar{x}), \tag{24}$$

$$\varphi^{(1)} = a_1 k'(\bar{x})v'(\bar{x}) + a_2 \frac{(k'(\bar{x}))^2}{k(\bar{x})} v'(\bar{x}) + a_3 k''(\bar{x})v(\bar{x}) + a_4 k(\bar{x})v''(\bar{x}) + \\ + b_1 q'(\bar{x})v(\bar{x}) + c_1 f'(\bar{x}), \tag{25}$$

where

$$\left.\begin{array}{l} a_0 = B_1^{(0)}[s] - A_1^{(0)}[s] - 1, \quad a_1 = B_1^{(1)}[s] - A_1^{(1)}[s], \\ a_2 = B_2^{(0)}[s] - A_2^{(0)}[s], \quad a_3 = 0{\cdot}5(B_1^{(0)}[s^2] - A_1^{(0)}[s^2]), \\ a_4 = 0{\cdot}5(B_1^{(0)}[s] + A_1^{(0)}[s]), \quad b_1 = -D^{(0)}[s], \quad c = F^{(0)}[s]. \end{array}\right\} \tag{26}$$

6. *The order of approximation.* We shall say that the homogeneous difference $L_h^{(k,q,f)}$ has nth order approximation of if the approximation error

$$\varphi(\bar{x}, v; h) = (L_k^{(k,q,f)}v)_{x=\bar{x}} - (L^{(k,q,f)})_{x=\bar{x}}$$

for this scheme is of the nth order of smallness in h:

$$\varphi(\bar{x}, v; h) = O(h^n)$$

for sufficiently smooth functions $v(x)$, $k(x)$, $q(x)$ and $f(x)$ (in particular for $k(x) \in C^{(n+1)}$, $q(x) \in C^{(n)}$, $f(x) \in C^{(n)}$, $v(x) \in C^{(m)}$, $m \geqslant n+2$).

From the series (23) it follows that the necessary and sufficient conditions (A.C.) for the scheme $L_k^{(k,q,f)}$ from the class $\Omega(n+1, n, n)$ to have nth order approximation are as follows for moments of the pattern functionals:

(1) for a scheme of the first order $(n = 1)$

$$B_1^{(0)}[s] - A_1^{(0)}[s] = 1, \quad B^{(0)}[1] = A^{(0)}[1] = D^{(0)}[1] = F^{(0)}[1] = 1 \tag{A.C.1}$$

(the normalisation condition of the pattern functionals is automatically satisfied for schemes of this class);

(2) for a scheme of the second order $(n = 2)$

$$\left.\begin{array}{l} B_1^{(0)}[s] = 0{\cdot}5, \quad A_1^{(0)}[s] = -0{\cdot}5, \quad B_1^{(0)}[s^2] = A_1^{(0)}[s^2], \\ B_2^{(0)}[s] = A_2^{(0)}[s], \\ B_1^{(1)}[s] = A_1^{(1)}[s], \quad D^{(0)}[s] = F^{(1)}[s] = 0, \\ A_1^{(1)}[1] = B_1^{(1)}[1] = D^{(1)}[1] = F^{(1)}[1] = 0 \end{array}\right\} \tag{A.C.2}$$

(the last four conditions are a consequence of the normalisation of the pattern functionals).

7. *Canonical schemes.* If the pattern functionals of the scheme $L_h^{(k,q,f)}$ do not depend on the parameter h, then we call them canonical functionals, and call the scheme a canonical scheme.

To each scheme $L_h^{(k,q,f)}$ of the initial type there corresponds a cannonical scheme $\bar{L}_h^{(k,q,f)}$ of the same rank

$$\bar{L}_h^{(k,q,f)} y_i = \bar{L}_h^{(k)} y_i - D_i^{(0,q)} \times y_i + F_0^{(0,f)},$$

where

$$\bar{L}_h^{(k)} y_i = \frac{1}{h^2} [B_i^{(0,k)} \Delta y_i - A_i^{(0,k)} \nabla y_i],$$

$$B_i^{(0,k)} = B^{(0)}[k(x_i + sh)], \qquad A_i^{(0,k)} = A^{(0)}[k(x_i + sh)],$$

$$D_i^{(0,q)} = D^{(0)}[q(x_i + sh)], \qquad F_i^{(0,f)} = F^{(0)}[f(x_i + sh)].$$

If $L_h^{(k,q,f)}$ has nth order approximation ($n = 1, 2$) then $\bar{L}_h^{(k,q,f)}$ also (from (A.C.1) and (A.C.2)) has the nth order approximation and these schemes are equivalent as far as approximation is concerned.

When discussing canonical schemes we shall omit the zero index in their pattern functionals $A[\psi]$, $B[\psi]$, $D[\psi]$ and $F[\psi]$. Then the conditions for second order approximation, for example, take the form:

$$B_1[s] = -A_1[s] = 0{\cdot}5, \qquad B_1[s^2] = A_1[s^2], \qquad D[s] = F[s] = 0. \qquad \text{(A.C.2}^{(0)}\text{)}$$

8. *Conservative difference schemes.* We consider the difference scheme $L_h^{(k)} = L_h^{(k,0,0)}$. Fixing $k(x)$, we obtain the difference operator

$$L_h y_i = \frac{1}{h^2} [B_i \Delta y_i - A_i \nabla y_i], \qquad 0 < i < N.$$

By analogy with the differential operator of the second order, we call the difference operator L_h self-conjugate if the expression $u_i L_h v_i - v_i L_h u_i$ can be put in the form of some difference $\Delta Q_i = Q_{i+1} - Q_i$ for any u_i, v_i at each point i. It is not difficult to show that the necessary and sufficient condition for the self-conjugacy of a difference operator L_h is the relation $B_i = A_{i+1}$ for all $1 \leqslant i \leqslant N-1$, so that we can write

$$L_h y_i = \frac{1}{h^2} \Delta(A_i \nabla y_i) \quad \text{or} \quad L_h y_i = \frac{1}{h^2} \Delta(A_i \Delta y_{i-1}). \tag{27}$$

We have the identity

$$u_i L_h v_i - v_i L_h u_i = \frac{1}{h^2} \Delta [A_i (u_{i-1} v_i - u_i v_{i-1})],$$

where u_i and v_i are arbitrary net functions.

This leads at once to Green's second difference formula

$$\sum_{i=1}^{N-1} (u_i L_h v_i - v_i L_h u_i) h = \frac{1}{h} [A_i (u_{i-1} v_i - v_{i-1} u_i)] \Big|_1^N.$$

We call the difference operator (27) a conservative operator. This term expresses the physical meaning of the difference equation $L_h y_i = -F_i$ which can be treated as the equation for the stationary temperature distribution y_i in the presence of heat sources.

Introducing the difference analogue of heat flow $w_i = -A_i \nabla y_i / h$ at the point $x = x_{i-1/2}$ and rewriting the equation $L_h y_i = -F_i$ in the form

$$w_{i+1} - w_i = -F_i h,$$

we see that it expresses the law of the conservation of heat on the interval $(x_{i-1/2}, x_{i+1/2})$ of length h. On the left we have the heat flow difference at the ends of the interval, and on the right we have the amount of heat which produced in the interval due to the sources.

Let us return now to the difference scheme

$$L_h^{(k)} y_i = \frac{1}{h^2}(B_i^{(h,k)} \Delta y_i - A_i^{(h,k)} \nabla y_i).$$

The difference scheme $L_h^{(k)}$ is said to be conservative if, for any function $k(x) \in Q^{(0)}$ the corresponding difference operator is conservative, i.e.

$$B_i^{(h,k)} = A_{i+1}^{(h,k)}$$

and, therefore

$$B^h[\psi(s)] = A^h[\psi(1+s)] \qquad (\psi(s) \in Q^{(0)}).$$

It follows that the functional $A^h[\psi(s)]$ is independent of the values of the function $\psi(s)$ for $0 < s \leqslant 1$, and the functional $B^h[\psi(s)]$ is independent of the values of $\psi(s)$ for $-1 \leqslant s < 0$.

If the scheme $L_h^{(k)}$ is conservative, then, clearly, the corresponding cannonical scheme is conservative too.

The difference scheme $L_h^{(k,q,f)}$ is said to be conservative if the scheme $L_h^{(k)}$ is conservative.

The following method, which we call the integro-interpolation method (I.I.M), can be used to obtain difference schemes when solving different physical problems. In place of the differential equation we write an integral relation expressing the conservation law (balance) for an elementary cell in the net. To substitute for the derivatives and integrals appearing in the balance equation we interpolate for the required function and the coefficients in the neighbourhood of a node. As a result we obtain a difference equation whose coefficients depend essentially on the character of the interpolation used both for the required function and for the coefficients of the initial equation.

We illustrate this method using the example of the equation $L^{(k,q,f)} u = 0$ and we show that it produces conservative schemes. We write the heat balance equation for the interval $(x_{i-1/2}, x_{i+1/2})$:

$$w_{i-1/2} - w_{i+1/2} - \int_{x_{i-1/2}}^{x_{i+1/2}} q(x) u(x) dx = -\int_{x_{i-1/2}}^{x_{i+1/2}} f(x) dx, \tag{28}$$

where $w(x) = -k(x)u'(x)$ is the heat flow at the point x. Therefore

$$u'(x) = -\frac{w(x)}{k(x)}.$$

Integrating with respect to u from x_{i-1} to x_i we shall have

$$u_i - u_{i-1} = -\int_{x_{i-1}}^{x_i} \frac{w(x)}{k(x)}\,dx. \tag{29}$$

Equations (28) and (29) are exact.

Assuming, for example, that $w(x) = \text{const} = w_{i-1/2}$ for $x_{i-1} < x < x_i$ we obtain

$$w_{i-1/2} \approx -A_i \frac{\nabla u_i}{h}, \quad A_i = \left[\frac{1}{h}\int_{x_{i-1}}^{x_i} \frac{dx}{k(x)}\right]^{-1}.$$

If $q(x) \equiv 0$, then we can obtain by this method the best canonical scheme $L_h^{(k,f)}$. To obtain the best canonical scheme $L_h^{(k,q,f)}$ we assume that $u = \text{const} = u_i$ for $x_{i-1/2} < x < x_{i+1/2}$, so that

$$\int_{x_{i-1/2}}^{x_{i+1/2}} q(x)u(x)\,dx \approx u_i \int_{x_{i-1/2}}^{x_{i+1/2}} q(x)\,dx.$$

It must be noted that the interpolations $w = \text{const}$ for $x \in (x_{i-1}, x_i)$ and $n = \text{const}$ for $x \in (x_{i-1/2}, x_{i+1/2})$ are not consistent with one another. However, the subsequent use of the same interpolation $w = w_{i-1/2}$ for $x \in (x_{i-1}, x_i)$ leads to a noticeable complication in the difference scheme without increasing its accuracy.

Other interpolations are also possible; for instance

$$\frac{w}{k} = \left(\frac{w}{k}\right)_{i-1/2} = \text{const} \quad \text{for } x \in (x_{i-1}, x_i), \text{ which gives } A_i = k_{i-1/2}.$$

With this method we obtain conservative schemes of the form

$$L_h^{(k,q,f)} y_i = \frac{1}{h}\Delta w_{i-1/2} - D_i^{(h,q)} w_{i-1/2} + F_i^{(h,f)},$$

where

$$w_{i-1/2} = -A_i^{(h,k)} \frac{\nabla y_i}{h}, \quad A_i^{(h,k)} = A^h[k(x_i + sh)], \quad -1 \leqslant s \leqslant 0,$$

$$D_i^{(h,q)} = D^h[q(x_i + sh)], \quad F_i^{(h,f)} = F^h[f(x_i + sh)], \quad -0{\cdot}5 \leqslant s \leqslant 0{\cdot}5,$$

D^h and F^h being linear functionals.

The integro-interpolation method was also used in [10] by G. I. Marchuk to construct discrete schemes (see Section 10) for open computing, in connection with the calculation of the critical dimensions of nuclear reactors.

We consider now the conservative scheme $L_h^{(k,q,f)}$ of $\mathfrak{L}(n+1, n, n)$ and take $k(x) \in C^{(n+1)}$, $q(x)$, $f(x) \in C^{(n)}$ $(n = 0, 1)$. If $n = 0$, then the approximation error

$$\varphi(x, u, h) = \varphi^{(0)}(x, u) + \rho(h).$$

If $n = 1$, then

$$\varphi(x, u, h) = \varphi^{(0)}(x, u) + O(h),$$

where $\varphi^{(0)}$ is given by formula (24).

We show that the condition for first order approximation $B_1^{(0)}[s] - A_1^{(0)}[s] = 1$ $(a_0 = 0)$ is satisfied for any conservative scheme of $\mathfrak{L}(1, 0, 0)$. For $B^h[\psi(s)] = A^h[\psi(1+s)]$ and, therefore $B_1^{(0)}[s] = A_1^{(0)}[(1+s)] = 1 + A_1^{(0)}[s]$. Therefore for a conservative scheme $\varphi^{(0)}(x, u) = 0$. We thus proved

LEMMA 4. *Any conservative scheme $L_h^{(k,q,f)}$ of the family $\mathfrak{L}(1, 0, 0)$ satisfies the conditions for first order approximation, and a conservative scheme of type $\mathfrak{L}(2, 1, 1)$ has first order approximation.*

If the conservative scheme $L_h^{(k,q,f)}$ is symmetric, then

$$A^h[\psi(-s)] = A^h[\psi(1+s)], \qquad D^h[\psi(-s)] = D^h[\psi(s)],$$

$$F^h[\psi(-s)] = F^h[\psi(s)].$$

This leads to

LEMMA 5. *Any conservative symmetric scheme $L_h^{(k,q,f)}$ of $\mathfrak{L}(2, 1, 1)$ satisfies* (A.C.2); *if such a scheme belongs to the family $\mathfrak{L}(3, 2, 2)$, then it has second order approximation.*

Let us now discuss the procedure of making the operator

$$L_h y_i = \frac{1}{h^2}(B_i \Delta y_i - A_i \Delta y_i)$$

conservative.

We multiply it by some function Λ_i and require that the operator $L_h^* = \Lambda_i L_h$ shall be conservative, i.e. $B_i \Lambda_i = A_{i+1}^*$, $A_i \Lambda_i = A_i^*$. From this we have

$$\Lambda_{i+1} = \Lambda_i (B_i / A_{i+1}) = \prod_{s=1}^{i} (B_s / A_{s+1}),$$

if we put $\Lambda_1 = 1$. Thus, we obtain the conservative operator

$$L_h^* y_i = \Lambda_i L_h y_i = \frac{1}{h^2} \Delta(A_i^* \nabla y_i), \qquad A_i^* = A_i \Lambda_i.$$

If the initial scheme $L_h^{(k)}$ is homogeneous, then the conservative scheme $L_h^{*(k)} = \Lambda_i L_h^{(k)}$ is not homogeneous.

9. *Linear schemes.* If $A^h[\psi]$ and $B^h[\psi]$ are linear functionals with respect to ψ, then we can call the scheme $L_h^{(k,q,f)}$ a k-linear scheme.

In addition to k-linear schemes, we can consider the so-called p-linear schemes

$$L_h^{(p)}y_i = \frac{1}{h^2}\left[\frac{1}{b_i^{(h,p)}}\Delta y_i - \frac{1}{a_i^{(h,p)}}\nabla y_i\right],$$

where $b_i^{(h,p)} = b^h[p(x_i+sh)]$, $a_i^{(h,p)} = a^h[p(x_i+sh)]$, with $a^h[\bar{p}(s)]$ and $b^h[\bar{p}(s)]$ being linear functionals, $\bar{p}(s) \in Q^{(0)}[-1, 1]$. This form corresponds to the differential operator

$$L^{(p)}u = \frac{d}{dx}\left[\frac{1}{p(x)}\frac{du}{dx}\right] \quad \left(p = \frac{1}{k}\right).$$

The study of linear schemes in the class of discontinuous coefficients is made easier by the fact that there is in $Q^{(0)}$ and integral representation for linear functionals using characteristic functions (see Section 11).

We note that the best canonical scheme $L_h^{(k)}$, obtained in § 4, for which

$$B_i = A_{i+1}, \quad A_i = \left[\frac{1}{h}\int_{x_{i-1}}^{x_i}\frac{dx}{k(x)}\right]^{-1} - \left[\frac{1}{h}\int_{x_{i-1}}^{x_i} p(x)dx\right]^{-1},$$

is a canonical conservative p-linear scheme in which

$$b[\bar{p}(s)] = a[\bar{p}(1+s)], \quad a[\bar{p}(s)] = \int_{-1}^{0}\bar{p}(s)\,ds.$$

10. *Discrete schemes.* If the pattern functionals $A^h[\psi(s)]$ and $B^h[\psi(s)]$ depend on the values of the function $\psi(s)$ on a discrete set of points, then they are called discrete functionals, and the corresponding scheme $L_h^{(k)}$ is a discrete scheme. When $\psi \in Q^{(0)}$ the discrete functional can depend not only on the values of ψ at separate points, but also on the left- and right-hand limit values of the functions ψ at these points.

As an example we consider the canonical discrete scheme $L_h^{(k)}$ whose coefficients are three-point discrete functionals:

$$A_i = f_1(k_{i-1}, k_i, k_{i+1}), \quad B_i = f_2(k_{i-1}, k_i, k_{i+1}) \quad (k_i = k(x_i)),$$

where $f_1(x, y, z)$ and $f_2(x, y, z)$ are some functions of three variables.

It follows from the condition $(\mathrm{II}^{(r)})$ for the pattern $A[\bar{k}(s)]$ and $B[\bar{k}(s)]$ that

$$f_1(x, y, z) = y\varphi(\bar{x}, \bar{z}), \quad f_2(x, y, z) = y\psi(\bar{x}, \bar{z}) \quad (\bar{x} = x/y, \bar{z} = z/y),$$

where $\varphi(1, 1) = 1$, $\psi(1, 1) = 1$.

The scheme mentioned in section 2 of the Introduction (Example 1) is discrete:

$$A_i = k_i - \frac{1}{4}(k_{i+1} - k_{i-1}), \quad B_i = k_i + \frac{1}{4}(k_{i+1} - k_{i-1}).$$

If the scheme is conservative, then

$$B_i = k_i\Psi\left(\frac{k_{i+1}}{k_i}\right) = A_{i+1}, \qquad A_i = k_i\Phi\left(\frac{k_{i-1}}{k_i}\right),$$

since, for conservative schemes $\varphi(\bar{x}, \bar{z}) = \Phi(\bar{x})$ is independent of $\bar{z}$, and $\psi(\bar{x}, \bar{z}) = \psi(\bar{z})$ is independent of $\bar{x}$.

It follows therefore that

$$\Psi(\xi) = \xi\Phi\left(\frac{1}{\xi}\right), \qquad \Phi(1) = 1\,.$$

If the scheme $L_h^{(k)}$ is also symmetric then $\bar{\psi}(\xi) = \Phi(\xi)$ and we obtain the functional equation

$$\Phi(\xi) = \xi\Phi\left(\frac{1}{\xi}\right), \qquad \Phi(1) = 1 \tag{29'}$$

for $\Phi(\xi)$. The general solution of this equation has the form $\Phi(\xi) = \sqrt{\xi}\,\omega(ln\,\xi)$ where $\omega(t)$ is an arbitrary even function satisfying the condition $\omega(0) = 1$*.

We give two examples of the function $\Phi(\xi)$:

1. $\Phi(\xi) = 0{\cdot}5(1+\xi)\,, \qquad A_i = 0{\cdot}5(k_{i-1}+k_i)\,, \qquad B_i = A_{i+1}\,.$

2. $\Phi(\xi) = \dfrac{2\xi}{1+\xi}\,, \qquad A_i = \dfrac{2k_{i-1}k_i}{k_{i-1}+k_i}\,, \qquad B_i = A_{i+1}\,.$

11. *Linear functionals in the class of discontinuous functions.* As we know, the linear functional $A[f]$ is defined by the conditions

$$1^\circ.\ A[f_1+f_2] = A[f_1+f_2]\,; \qquad 2^\circ.\ |A[f]| \leqslant M\sup|f|\,. \tag{30}$$

Consider the linear functional $A[f]$, defined for piece-wise continuous functions on the segment $[a, b]$.

Due to the fact that the representation of a linear functional in the class $C^{(0)}$ $[a, b]$ using Stieltjes's integral

$$A[f] = \int_a^b f(s)\,d\alpha(s) \qquad \text{(Riesz's theorem)}$$

is not continued uniquely in the class $Q^{(0)}$ we must find a representation for $A[f]$ in $Q^{(0)}[a, b]$. Such a representation is given in [12]. It is proved (Theorem 1) that the linear functional $A[f(s)]$ where $f(s) \in Q^{(0)}[a, b]$ is uniquely defined by the two characteristic functions:

$$\alpha(\lambda) = A[\eta_\lambda(s)], \qquad \sigma(\lambda) = A[\pi_\lambda(s)], \tag{31}$$

where

$$\eta_\lambda(s) = \begin{cases} 1, & s < \lambda, \\ 0, & s \geqslant \lambda, \end{cases} \qquad \pi_\lambda(s) = \begin{cases} 1, & s = \lambda, \\ 0, & s \neq \lambda. \end{cases} \tag{32}$$

* M. V. Maslennikov has pointed out the existence of a general solution for equation (29′).

If $\sigma(\lambda) \equiv 0$ for $\lambda \in [a, b]$, then the functional is said to be regular; while, if $\alpha(\lambda) \equiv 0$ for $\lambda \in [a, b]$ it is called a point functional.

We give here the properties of linear functionals and their characteristic functions which we shall heed below; for the proofs reference should be made to [12].

(1) There exists not more than a denumerable number of points $\zeta_1, \zeta_2, \ldots, \zeta_j, \ldots$, at which $\sigma(\zeta_j) \neq 0$, where

$$\sum_{j=1}^{\infty} |\sigma(\zeta_j)| < M .$$

(2) Any linear functional A can be put in the form of the sum of a reqular linear functional $\bar{A}$ and a point linear functional A^*

$$A[f] = \bar{A}[f] + A^*[f],$$

where

$$A^*[f] = \sum_{j=1}^{\infty} \sigma(\zeta_j) f(\zeta_j) .$$

The regular functional $\bar{A}[f]$ is wholly defined by the characteristic function

$$\bar{\alpha}(\lambda) = \bar{A}[\eta_\lambda(s)] = \alpha(\lambda) - \sum_{\zeta_j < \lambda} \sigma(\zeta_j) .$$

(3) (a) The function $\bar{\alpha}(\lambda)$ is of bounded variation;

(b) there exists not more than a denumerable number of points $\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots$, at which $\bar{\alpha}_-(\lambda_i) \neq \bar{\alpha}(\lambda_i)$ or $\bar{\alpha}(\lambda_i) \neq \bar{\alpha}_+(\lambda_i)$, $s = 1, 2, \ldots$;

(c) the function

$$\bar{\bar{\alpha}}(\lambda) = \bar{\alpha}_-(\lambda) - \sum_{\lambda_i < \lambda} [\bar{\alpha}_+(\lambda_i) - \bar{\alpha}_-(\lambda_i)]$$

is a continuous function on $[a, b]$.

(4) We have

THEOREM 1. *Any linear functional $A[f]$, defined in the class $Q^{(0)}[a, b]$, can be put in the form*

$$A[f] = \int_a^b f(s) d\bar{\bar{\alpha}}(s) + \sum_{i=1}^{\infty} \{f_+(\lambda_i)[\bar{\alpha}_+(\lambda_i) - \bar{\alpha}(\lambda_i)] + f_-(\lambda_i)[\bar{\alpha}(\lambda_i) - \bar{\alpha}_-(\lambda_i)]\} + \\ + \sum_{j=1}^{\infty} \sigma(\zeta_j) f(\zeta_j) . \qquad (33)$$

If it follows from $f \geq 0$ that $A[f] \geq 0$, then the linear functional $A[f]$ is said to be non-negative.

THEOREM 2. *In order that the linear functional* $A[f]$ *shall be non-negative it is necessary and sufficient that these two conditions are satisfied*:

(1) *the characteristic function* $\bar{\alpha}(\lambda)$ *of the regular part* $\bar{A}$ *of the functional* A *is a non-decreasing function*;

(2) $\sigma(\zeta_j) \geqslant 0$ *for all* $j = 1, 2, \ldots$

That these conditions are sufficient follows from (33). We prove that they are necessary. Let $A[f]$ be a non-decreasing functional. Taking

$$f_j(s) = \pi_{\zeta_j}(s) \geqslant 0,$$

we shall have $\sigma(\zeta_j) = A[f_j(s)] \geqslant 0$ for any $j = 1, 2, \ldots$

Introducing the function

$$\bar{f}_\lambda(s) = \eta_\lambda(s) - \sum_{\zeta_j < \lambda} \pi_{\zeta_j}(s) \geqslant 0,$$

we note that

$$A[\bar{f}_\lambda(s)] = \alpha(\lambda) - \sum_{\zeta_j < \lambda} \sigma(\zeta_j) = \bar{\alpha}(\lambda). \tag{34}$$

Let λ_1 and $\lambda_2 \geqslant \lambda_1$ be any points of the segment $[a, b]$.

From the inequality

$$\bar{f}_{\lambda_2}(s) - \bar{f}_{\lambda_1}(s) \geqslant 0,$$

and formula (34) it follows that

$$A[\bar{f}_{\lambda_2}(s) - \bar{f}_{\lambda_1}(s)] = \bar{\alpha}(\lambda_2) - \bar{\alpha}(\lambda_1) \geqslant 0 \quad \text{when} \quad \lambda_2 \geqslant \lambda_1,$$

i.e. $\bar{\alpha}(\lambda)$ is a non-decreasing function.

In other words, the necessary and sufficient conditions for the linear functional $A[f]$ to be non-negative are the conditions for its regular and point components $\bar{A}[f]$ and $A^*[f]$ to be non-negative.

The homogeneous functional of the first degree, $A[f]$, is completely determined by its first differential $A_1[f, \varphi]$, which is a linear functional with respect to it second argument φ; we have the equality

$$A[f] = A_1[f, f] \quad \text{(Lemma 1)}. \tag{35}$$

§ 2. HOMOGENEOUS DIFFERENCE SCHEMES IN THE CLASS OF SMOOTH COEFFICIENTS

In this paragraph we show that the order of approximation of a homogeneous scheme from the initial family $\mathcal{L}(n_1, n_2, n_3)$ in the class of sufficiently smooth coefficients is the same as the order of accuracy of this scheme.

1. *The accuracy of difference schemes in the class of smooth coefficients.* In § 1, when considering the question of the accuracy of the solution y_i^h of the difference boundary problem (II) with respect to the solution $u = u(x)$ of the initial problem (I) we obtained the following conditions for the net function $z_i^h = y_i^h - u(x)$:

$$L_h^{(k,q)} z_i^h = -\varphi_i^h, \quad 0 < i < N, \quad z_0^h = 0, \quad z_N^h = 0, \tag{III}$$

where

$$L_h^{(k,q)} z_i^h = \frac{1}{h^2}[B_i^{(h,k)}\Delta z_i^h - A_i^{(h,k)} \nabla z_i^h] - D_i^{(h,q)} z_i^h \tag{1}$$

is the initial scheme and

$$\varphi_i^h = L_h^{(k,q,f)} u_i - (L^{(k,q,f)} u)_i \tag{2}$$

is the approximation error of the scheme $L_h^{(k,q,f)}$ over the solution $u(x)$ of the boundary problem (I).

Let R_h be an operator giving the solution of problem (III):

$$z_i^h = R_h \varphi_i^h. \tag{3}$$

We introduce the norm for the net function z_i^h:

$$\| z_i^h \|_1 = \max_{0<i<N} | z_i^h | \tag{4}$$

and some norm $\|\varphi^h\|_2$ for the function φ_i^h. If, for fixed k and q the norm of the operator R_h is uniformly bounded with respect to h:

$$\| z_i^h \|_1 = \| R^h \varphi_i^h \|_1 \leqslant M_h^{(k,q)} \| \varphi_i^h \|_\sigma, \qquad M_h^{(k,q)} \leqslant M^{(k,q)}, \tag{5}$$

then the uniform convergence (see [6]) of the solutions of the difference boundary problem (II) to the solution of problem (I) will follow from the smallness of the approximation error according to the norm $\| \ \|_\sigma$.

In considering the convergence of difference schemes in the class of smooth coefficients the same norm can be taken for φ_i^h and z_i^h:

$$\| \varphi_i^h \|_\sigma = \| \varphi_i^h \|_1 = \max_{0<i<N} | \varphi_i^h | \qquad (\sigma = 1).$$

In Section 3 we consider Green's difference function for problem (III) and show that the operators R_h giving a solution of (III) are uniformly bounded with respect to h for any scheme $L_h^{(k,q)}$ (from the class $\mathfrak{L}(2, 1)$), if $k(x) \in C^{(1)}$ and $q(x) \in C^{(0)}$ (Lemma 2).

If $k(x) \in C^{(n+1)}$, $q(x) \in C^{(n)}$, $f(x) \in C^{(n)}$ $(n = 1, 2)$, and the scheme $L_h^{(k,q,f)}$ from the class $\mathfrak{L}(n+1, n, n)$ has nth order approximation, i.e. satisfies the conditions (A.C.1), $n = 1$ or (A.C.2), $n = 2$ (see § 1, Section 6) then we have the uniform estimate

$$\| \varphi^h \|_1 = O(h^n) \qquad \text{or} \qquad \| \varphi^h \|_1 \leqslant M \times h^n, \tag{6}$$

for the error φ_i^h where M is a positive constant depending on the choice of k, q, f and independent of h.

This, together with Lemma 2, leads to

LEMMA 3. *If the initial scheme $L_h^{(k,q,f)}$ of $\mathfrak{L}(n+1, n, n)$ has* nth *($n = 1, 2$) order approximation, then the solution of the boundary problem* (II) *has* nth *order accuracy in the class $k(x) \in C^{(n+1)}$, $q(x) \in C^{(n)}$, $f(x) \in C^{(n)}$.*

When discussing the accuracy of the solution of the boundary problem (II) in the class of discontinuous coefficients we require that the operator R_h shall be bounded according to the norm

$$\|\varphi_i^h\|_2 = \sum_{i=1}^{N-1} h\left|\sum_{s=1}^{i} h\varphi_s^h\right|. \tag{7}$$

It will be shown in Section 3 that the operators R_h are uniformly bounded with respect to h in this norm.

2. *Green's difference function.* We come now to the question of the existence and boundedness of the operator R_h, defined with the use of Green's difference function.

We consider the difference boundary problem

$$L_h z_i^h = -\varphi_i, \quad 0 < i < N, \quad z_0^h = 0, \quad z_N^h = 0, \tag{8}$$

where L_h is a difference operator defined by the expression

$$L_h z_i = \frac{1}{h^2}(B_i^h \Delta z_i - A_i^h \nabla z_i) - D_i^h z_i. \tag{9}$$

We shall assume that the coefficients of the operator L_h satisfy the conditions

$$0 < M_1 \leqslant A_i^h \leqslant M_2, \quad 0 < M_1 \leqslant B_i^h \leqslant M_2, \quad 0 \leqslant D_i^h \leqslant M_3 \quad (0 < i < N), \tag{α}$$

where M_1, M_2 and M_3 are positive constants independent of h.

The solution of this problem can be put in the form

$$z_i^h = R_h \varphi_i^h = \sum_{j=1}^{N-1} G_{ij}\varphi_i^h h \tag{10}$$

with the help of Green's difference function. In particular, if the boundary conditions are non-homogeneous, then

$$z_i^h = \sum_{j=1}^{N-1} G_{ij}\varphi_j^h h - \frac{A_1^h G_{i1}}{h} z_0^h + \frac{B_{N-1}^h G_{i,N-1}}{h} z_N^h. \tag{11}$$

We define Green's difference function G_{ij} of the problem (8) using the conditions:

(a) G_{ij} satisfies the equation

$$L_h G_{ij} = -\frac{\delta_{ij}}{h} \quad \text{with} \quad 0 < i < N, \quad \text{where} \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j'. \end{cases} \tag{12}$$

for variable i and fixed j $(0 \leqslant j \leqslant N)$.

(b) G_{ij} satisfies the homogeneous boundary conditions

$$G_{1j} = 0, \quad G_{Nj} = 0. \tag{13}$$

We show that the function G_{ij} exists if conditions (α) are satisfied. We consider two cases.

A. The difference operator L_h is conservative, i.e. $B_i^h = A_{i+1}^h$ and $L_h z_i = \frac{1}{h^2}\Delta(A_i^h \nabla z_i) - D_i^h z_i$.

By analogy with differential equations, we shall look for G_{ij} in the form

$$G_{ij} = \begin{cases} a_j v_i, & i < j, \\ b_j w_i, & i > j, \end{cases} \tag{14}$$

where a_j and b_j are factors to be determined, and v_i and w_i are the solutions of the homogeneous equation $L_h z_i = 0$ satisfying the conditions

$$L_h v_i = 0, \quad 0 < i < N, \quad v_0 = 0, \quad \frac{A_1^h \Delta v_0}{h} = 1 \quad \text{or} \quad v_j = \frac{h}{A_1^h}; \tag{15}$$

$$L_h w_i = 0, \quad 0 < i < N, \quad w_N = 0, \quad -A_N^h \frac{\Delta w_{N-1}}{h} = 1 \quad \text{or} \quad w_{N-1} = \frac{h}{A_N^h}. \tag{15'}$$

Using Green's second difference formula (see Section 8, § 1) for the functions v_i and w_i we see that

$$v_N = w_0. \tag{16}$$

From the conditions (15) we obtain

$$A_i \frac{\nabla v_i}{h} = 1 + \sum_{s=1}^{i-1} D_s^h v_s h. \tag{17}$$

From this, and from the conditions $D_s^h \geqslant 0$, $v_1 = \frac{h}{A_1^h} > 0$ it follows that $v_i > 0$ when $i > 0$, or more precisely,

$$v_i \geqslant \sum_{k=1}^{i} \frac{h}{A_k^h} \geqslant \frac{x_i}{M_2}, \quad v_N \geqslant \frac{1}{M_2},$$

and, similarly, $w_i \geqslant \frac{1-x_i}{M_2}$.

Thus, Green's function $G_{ij} \geqslant 0$ $(0 \leqslant i,\ j \leqslant N)$.

From the condition $a_j v_j = b_j w_j$ and from equation (12) for $i = j$ we find that

$$a_j = \frac{w_j}{\Delta}, \quad b_j = \frac{v_j}{\Delta},$$

where

$$\Delta = \frac{1}{h} A_i(v_i w_{i-1} - v_{i-1} w_i) = \text{const}.$$

From conditions (15) and (15′) for the functions v_i and w_i it follows that

$$\Delta = v_N = w_0.$$

As a result we obtain the following expression for Green's difference function:

$$G_{ij} = \begin{cases} v_i w_j / v_N, & i < j, \\ w_i v_j / v_N, & i > j. \end{cases} \tag{18}$$

We obtain at once the symmetry property of Green's function: $G_{ij} = G_{ji}$.

B. If the difference operator L_h is not conservative, i.e.

$$L_h y_i = \frac{1}{h^2}(B_i^h \Delta y_i - A_i^h \nabla y_i) - D_i^h y_i,$$

then, multiplying it by the factor

$$\Lambda_i = \prod_{s=1}^{i-1} \frac{B_s^h}{A_{s+1}^h},$$

according to Section 6 of § 1 we obtain the conservative operator

$$L_h^* y_i = \frac{1}{h_2} \Delta(A_i^{*h} \nabla y_i) - D_i^{*h} y_i,$$

where $A_i^{*h} = A_i^{*h}\Lambda_i$. To the operator L_h^* corresponds Green's difference function G_{ij}^*, which is constructed as in case **A**.

From the equations for G_{ij} and G_{ij}^* we see that

$$G_{ij} = \Lambda_j G_{ij}^*.$$

3. *Estimates of Green's difference function.* When deriving uniform estimates (uniform with respect to h) for Green's function, we shall assume that, together with the condition (α), the inequality

$$e^{-bh} \leqslant \frac{B_i^h}{A_{i+1}^h} \leqslant e^{bh}, \qquad 0 < i < N-1, \tag{β}$$

is satisfied, where b is a positive constant independent of the parameter h.

Consider first the case **A**.

It follows from formula (18), and the inequality $v_i \geqslant x_i/M_2$, $w_j \geqslant (1-x_j)/M_2$ that

$$\frac{\psi_{ij}}{M_2 v_N} \leqslant G_{ij} \leqslant v_N, \tag{19}$$

where $\psi_{ij} = \begin{cases} x_i(1-x_j) \\ x_j(1-x_i) \end{cases}$ when $\begin{matrix} i < j, \\ i > j. \end{matrix}$

We see from this that to obtain a two-sided estimate for Green's function it is sufficient to find an upper estimate for v_N. To do this we need

LEMMA 1. *Let v_i be a solution of problem* (15), *and $\bar{v}_i$ a solution of the problem*

$$\bar{L}_h \bar{v}_i = 0, \quad \bar{v}_0 = 0, \quad \bar{A}_1 \bar{v}_1 = h, \quad \bar{L}_h \bar{v}_i = \frac{1}{h^2}\Delta(\bar{A}_i \nabla \bar{v}_i) - \bar{D}_i \bar{v}_i.$$

If the conditions $\bar{A}_i \leqslant A_i$, $\bar{D}_i \geqslant D_i$, together with the conditions (α), *hold for the coefficients of the operators L_h and $\bar{L}_h$, then $\bar{v}_i \geqslant v_i$.*

For, the function $z_i = \bar{v}_i - v_i$ satisfies the conditions

$$\bar{L}_h z_i = \frac{1}{h^2} \Delta[(A_i - \bar{A}_i)\nabla v_i] + (\bar{D}_i - D_i) v_i, \quad z_0 = 0, \quad z_1 = h\left(\frac{1}{\bar{A}} - \frac{1}{A_1}\right) \geqslant 0.$$

We obtain

$$\frac{\bar{A}_i \nabla z_i}{h} = \left(1 - \frac{\bar{A}_i}{A_i}\right)\left(1 + \sum_{s=1}^{i-1} D_s v_s h\right) + \sum_{s=1}^{i-1} D_s v_s h + \sum_{s=1}^{i-1} (\bar{D}_s - D_s) v_s h \geqslant 0,$$

i.e. $z_i \geqslant z_{i-1} \geqslant z_1 \geqslant 0$ or $\bar{v}_i \geqslant v_i$.

Putting $\bar{A}_i = M_1$, $\bar{D}_i = M_3$ we obtain an equation for $\bar{v}_i$ with constant coefficients

$$\Delta^2 \bar{v}_{i-1} - \varkappa^2 h^2 \bar{v}_i = 0, \quad \varkappa^2 = M_3/M_1,$$

whose solution, when $\bar{v}_0 = 0$, $\bar{v}_1 = \frac{h}{\bar{A}_1}$, takes the form

$$\bar{v}_i = \frac{\operatorname{sh} \omega x_i}{M_1 \varkappa \sqrt{1 + \left(\frac{\varkappa h}{2}\right)^2}},$$

where ω is the root of the equation $\operatorname{sh} \frac{\omega h}{2} = \frac{\varkappa h}{2}$ such that $\sin \omega \leqslant \sin \varkappa$.

Therefore, we obtain the estimate

$$v_N \leqslant \frac{\operatorname{sh} \varkappa}{M_1 \varkappa} = \frac{\operatorname{sh} \sqrt{\frac{M_3}{M_1}}}{\sqrt{M_1 M_3}} \tag{20}$$

for v_N, for any values of $h > 0$.

We arrive at the following upper and lower bounds for Green's function:

$$2g_0 \psi_{ij} = \frac{\psi_{ij} \sqrt{M_1 M_3}}{M_2^2 \operatorname{sh} \sqrt{\frac{M_3}{M_1}}} \leqslant G_{ij} \leqslant \frac{\operatorname{sh} \sqrt{\frac{M_3}{M_1}}}{\sqrt{M_1 M_3}}. \tag{21}$$

If $D_i^h = 0$ then formula (21) gives

$$\frac{M_1}{M_2^2} \psi_{ij} \leqslant G_{ij}^0 \leqslant \frac{1}{M_1}, \tag{22}$$

for the corresponding Green's function G_{ij}^0 ($M_3 = 0$).

From the inequality $G_{ij} \leqslant G_{ij}^0$ (with the condition $D_i^h \geqslant 0$) it follows that we can use the simpler upper bound for Green's function

$$G_{ij} \leqslant \frac{1}{M_1} = g_1. \tag{23}$$

For the first difference ratio of Green's function we have

$$\frac{G_{i,j+1}-G_{ij}}{h}=\begin{cases}\dfrac{v_i}{v_N}\dfrac{\Delta w_j}{h}, & i<j,\\[2mm] \dfrac{w_i}{v_N}\dfrac{\Delta v_j}{h}, & i>j.\end{cases}$$

From the inequality

$$\frac{\nabla v_j}{h}=\frac{1}{A_j}+\frac{1}{A_j}\sum_{s=1}^{j-1}D_s v_s h\leqslant\frac{1}{M_1}+\frac{M_3}{M_1}v_N\leqslant M_5(M_1,M_3),$$

we obtain

$$\left|\frac{G_{i,j+1}-G_{ij}}{h}\right|\leqslant M_5, \tag{24}$$

where $M_5=M_5(M_1, M_3)$ is a positive constant, depending only on M_1 and M_3.

Taking, for example, $i=j=i_0=\left[\frac{N}{2}\right]$, we obtain from (21)

$$G_{i_0 i_0}\geqslant g_0=\frac{\sqrt{M_1M_3}}{2M_2^2\operatorname{sh}\sqrt{M_3/M_1}}. \tag{25}$$

If the operator L_h is not conservative, and satisfies the conditions (α) and (β), then we shall have

$$0<M_1'\leqslant A_i^*\leqslant M_2',\qquad 0\leqslant D_i^*\leqslant M_3', \tag{26}$$

for the coefficients of the conservative operator $L_h^*=\Lambda_iL_h$, where $M_1'=M_1\bar{e}^b$, $M_2'=M_2e^b$, $M_3'=M_3e^b$.

We have the two-sided estimate

$$\bar{e}^b\leqslant\Lambda_i\leqslant e^b,\qquad 0<i<N \tag{27}$$

for the factor Λ_i.

To estimate G_{ij}^* we can use (21), replacing the constants M^k ($k=1, 2, 3$) by the constants M_k'. We obtain an estimate for G_{ij} if we use relation (27) and the formula

$$G_{ij}=\Lambda_jG_{ij}^*.$$

We return now to the formula

$$z_i^h=\sum_{j=1}^{N-1}G_{ij}\varphi_j^hh.$$

Using (23) we have

$$\|z^h\|_1=\max_{0<i<N}|z_i^h|\leqslant\frac{1}{M_1'}\sum_{j=1}^{N-1}|\varphi_j^h|h. \tag{28}$$

From this we have

$$\|z^h\|_1\leqslant\frac{1}{M_1'}\|\varphi^h\|_\sigma,\qquad \sigma=1, 3. \tag{29}$$

Thus, the constant $\frac{1}{M'}$ also gives a uniform estimate of the operator R_h.

LEMMA 2. *The operators R_h, which give the solution of problem* (III) ($z^h = R_h\varphi^h$) *for any scheme $L_h^{(k,q)}$ of the class $\mathfrak{L}(2, 1)$ and for $k(x) \in C^{(1)}$, $q, f \in C^{(0)}$ are uniformly bounded both according to the norm $\| \ \|_1$ and according to the norm $\| \ \|_2$, i.e.*

$$\| z^h \|_1 \leqslant g_\sigma \| \varphi^h \|_\sigma, \quad \sigma = 1, 2,$$

where g_1 and g_2 are positive constants depending only on M_1, M_2, M_3 and b

$$\left(\| \varphi^h \|_1 = \max_{0<i<N} |\varphi_i^h|, \quad \| \varphi^h \|_2 = \sum_{i=1}^{N-1} h \left| \sum_{s=1}^{i} \varphi_s h \right| \right).$$

Thus conditions (α) will be satisfied if

$$0 < M_1 \leqslant k(x) \leqslant M_2, \quad 0 \leqslant q(x) \leqslant M_3,$$

since the pattern functionals of the scheme are normalised and are non-decreasing.

If $k(x) \in C^{(1)}$, then $A_i^{(h,k)} = k_i + hk_1' A_1^{(0)}[s] + h\rho(h)$,

$$B_i^{(h,k)} = k_i + hk_1' B_1^{(0)}[s] + h\rho(h),$$

$$B_i^{(h,k)} / A_{i+1}^{(h,k)} = 1 + h\left(\frac{k'}{k}\right)_i \cdot a_0 + h\rho(h), \quad a_0 = B_1^{(0)}[s] - A_1^{(0)}[s] - 1.$$

We see from this that we can always choose the constant b so that condition (β) is satisfied. For example

$$b = (1 + |a_0|) \max_{0 \leqslant x \leqslant 1} \left| \frac{k'}{k} \right|.$$

When considering the convergence of our schemes in the class of discontinuous coefficients we need a more exact estimate, as well as the estimate (29). Putting z_i in the form

$$z_i^h = \sum_{j=1}^{N-1} G_{ij} \Delta \left(\sum_{s=1}^{j-1} \varphi_s^h h \right)$$

and using the identity $k_j \Delta v_j = -v_{j+1} \Delta u_j + \Delta(u_j v_j)$ we obtain

$$z_i^h = -\sum_{j=1}^{N-1} \left(\frac{G_{i,j+1} - G_{ij}}{h} \right) \left(\sum_{s=1}^{j} \varphi_s^h h \right) h.$$

From this follows the inequality

$$\| z^h \|_1 \leqslant \max_{0 \leqslant i, j \leqslant N-1} \left| \frac{G_{i,j+1} - G_{ij}}{h} \right| \cdot \sum_{s=1}^{N-1} h \left| \sum_{s=1}^{j} \varphi_s^h h \right|,$$

which takes the form

$$\| z^h \|_1 \leqslant M_5' \| \varphi^h \|_2, \quad \text{where} \quad M_5' = e^b M_5(M_1', M_3'), \tag{30}$$

if we use the uniform estimate (24) for the difference ratio of Green's function.

NOTE. *If condition* (β) *is satisfied everywhere apart from a finite number of points* $j = 1, 2, \ldots, j_0$ *then all the estimates obtained above are still valid if, instead of* b, *we introduce a new constant* b_1 *using the condition* $b_1 = b + j_0 \ln \frac{M_z}{M_1}$; *then* $\bar{e}^{b_1} \leqslant \Lambda_i \leqslant e^{b_1}$, $M_1' = \bar{e}^b M_1$, $M_2' = e^b M_2$ *and so on.*

4. *The order of approximation of convergent schemes.* We come now to consider the relation between the order of approximation and the order of accuracy of a scheme for problem (II). In Section 1 we stated Lemma 3. Its proof follows from inequalities (29) and (5).

Can we make the reverse statement? Is the order of approximation of a scheme determined by its order of accuracy? The answer to this question is given by

LEMMA 4. *If the difference scheme* $L_h^{(k,q,f)}$ *of the family* $\mathfrak{L}(n+1, n, n)$ *has* nth *order accuracy in the class of sufficiently smooth coefficients*

$$k(x) \in C^{(m_k)}, \quad q(x) \in C^{(m_q)}, \quad f(x) \in C^{(m_f)}, \quad m_k \geqslant n+1, \quad m_q \geqslant n, \quad m_f \geqslant n,$$

then it has nth *order approximation* ($n = 1, 2$).

Thus, suppose $z_i = O(h^n)$ for any coefficients of the given class. We show that then the conditions for nth order approximation must be satisfied (see (A.C.1) and (A.C.2) in Section 6, § 1). We can use the inequality

$$|z_{i_0}| \geqslant g_0 \Big|\sum_{j=1}^{N-1} \psi_{i_0 i}\varphi_j h\Big| > \frac{1}{4} g_0 .$$

In particular, if $\varphi_j = \bar{\varphi}\cdot h^m + O(h^{m+1})$ where $\bar{\varphi} = \text{const}$, $m \geqslant 0$ then it follows that

$$|\bar{\varphi}| \leqslant \frac{|z_{i_0}|}{\bar{g}_0 h^m} + O(h), \qquad 0 < \bar{g}_0 < g_0 \sum_{j=1}^{N-1} \psi_{i_0 j} h . \tag{31}$$

In Section 5, § 1 we obtained the expansion in powers of h

$$\varphi = \varphi^{(0)} + h\varphi^{(1)} + O(h^2),$$

for the approximation error $\varphi(x, u, h)$, where $\varphi^{(0)}$ and $\varphi^{(1)}$ were determined from formulae (25) and (26), § 1 in terms of the moment of the differentials of the pattern functionals, the functions $k(x)$, $q(x)$, and $f(x)$ and their derivatives.

We choose the functions $u(x)$, $k(x)$, $q(x)$ and $f(x)$ so that $\varphi^{(0)}$ and $\varphi^{(1)}$ are constant.

Let $n = 1$. Putting $k(x) = e^x$, $u(x) = e^{-x}$ and using (31) for $m = 0$, we find that $\varphi^{(0)} = -a_0 = O(h)$, i.e. $a_0 = 0$.

Consider now a scheme of $\mathfrak{L}(3, 2, 2)$, $n = 2$. We have $\varphi = \varphi^{(1)}h + O(h^2)$ since $\varphi^{(0)} = 0$ ($m = 1$). To show that the relations $a_j = 0$, $j = 1, 2, 3, 4$ are valid we give the following examples:

(1) $k(x) = 1, \quad f(x) = x, \quad q(x) = 0, \quad \varphi^{(1)} = F_1^{(0)}[s]$

(the inequality (31) gives $\varphi^{(1)} = O(h)$ i.e. $F_1^{(0)}[s] = 0$);

(2) $k(x) = 1, \quad f(x) = 1 - x, \quad q(x) = 1 - x, \quad u(x) = 1, \quad \varphi^{(1)} = D_1^{(0)}[s] = 0$;

(3) $k(x) = e^x, \quad u(x) = e^{-x}, \quad \varphi^{(1)} = -a_1 - a_2 - a_3 + a_4 = 0$;

(4) $k(x) = e^x$, $u(x) = e^x$, $\varphi^{(1)} = -a_1 + a_2 + a_3 - a_4 = 0$, $a_1 = 0$, $a_4 = a_2 + a_3$; therefore $\varphi^{(1)}$ can be rewritten in the form

$$\varphi^{(1)} = a_3\left(\frac{k'}{k}\right)' . ku' - a_4\left(\frac{k'}{k}\right) . f . u,$$

if we use the equation $ku'' + k'u' = -f$;

(5) $k(x) = e^x$, $f(x) = -1$, $q(x) = 0$, $\varphi^{(1)} = -a_4 = 0$;

(6) $\left(\frac{k'}{k}\right) . ku' = 1$, $\varphi^{(1)} = a_3 = 0$.

This proves the lemma.

From Lemmas 3 and 4 we obtain the following:

THEOREM 1. *For the difference scheme* $L_h^{(k,q,f)}$ *of the class* $\mathfrak{L}(n+1, n, n)$ *for any coefficients* $k(x) \in C^{(n+1)}$, $q(x) \in C^{(n)}$ $f(x) \in C^{(n)}$ *(satisfying conditions* (α) *and* (β), *see Section* 1, § 1) *to have* nth *order accuracy it is necessary and suffcient that it has* nth *order approximation* ($n = 1, 2$).

§ 3. THE NECESSARY CONDITIONS FOR CONVERGENCE IN THE CLASS OF DISCONTINUOUS COEFFICIENTS

In this paragraph we establish the necessary conditions for the convergence of the difference scheme $L_h^{(k,q,f)}$ in the class of discontinuous coefficients.

1. *The approximation error in the neighbourhood of a point of discontinuity of the coefficients.* The error $z_i^h = y_i^h - u(x_i)$ of the solution y_i^h of the difference problem (II) relative to the solution $u(x)$ of problem (I) is determined, as we have seen, from the conditions

$$L_h^{(k,q,f)} z_i^h = -\varphi_i^h, \quad 0 < i < N, \quad z_0^h = 0, \quad z_N^h = 0. \tag{III}$$

The right-hand side of the equation $\varphi_i^h = L_h^{(k,q,f)} u_i - (L^{(k,q,f)} u)_i$ is the approximation error of the scheme $L_h^{(k,q,f)}$ over the solution $u = u(x)$ of problem (I).

If the coefficient $k(x)$ of the equation has a first order discontinuity at some point $x = \xi$, then in the neighbourhood of this point the scheme $L_h^{(k,q,f)}$ does not approximate to the differential operator $L^{(k,q,f)}$.

The position of the point $x = \xi$ on the difference net $S_h\{x_0 = 0, \ldots, x_i = xh, \ldots, x_N = Nh = 1\}$ is defined by the two numbers n and θ:

$$\xi = x_n + \theta h, \quad 0 < \theta < 1, \quad x_n = nh. \tag{1}$$

It is obvious that n and θ are functions of the step h or the number N:

$$n = n(h), \quad \theta = \theta(h). \tag{2}$$

At the point $x = \xi$ the solution $u = u(x)$ of problem (I) satisfies the conjugacy conditions

$$u(\xi - 0) = u(\xi + 0) = u(\xi), \quad k_- u'_- = k_+ u'_+ \quad \big(k_- = k(\xi - 0),\ k_+ = k(\xi + 0)\big). \tag{3}$$

We shall consider the initial scheme $L_h^{(k,q,f)}$ of the family of schemes $\mathcal{L}(2, 1, 1)$ of first order approximation with the coefficients

$$A_i^{(h,k)} = A^h[k(x_i+sh)], \qquad B_i^{(h,k)} = B^h[k(x_i+sh)],$$
$$D_i^{(h,q)} = D^h[q(x_i+sh)], \qquad F_i^{(h,f)} = F^h[f(x_i+sh)],$$

where D^h and F^h are linear functionals. We now assume that $k(x)\in Q^{(2)}$, $q(x)\in Q^{(1)}$, $f(x)\in Q^{(1)}$. Since the scheme is three-point

$$\varphi_i^h = O(h), \quad i\neq n, \quad i\neq n+1\,. \tag{4}$$

For the estimates of φ_n^h and φ_{n+1}^h we expand $u(x)$ in the neighbourhood of the point $x=\xi$:

$$u_{n+j} = u(\xi)+(j-\theta)hu_n'+\frac{(j-\theta)^2}{2!}h^2.\,u_n''+O(h^3) \qquad (j=2,\,1,\,0,\,-1).$$

In future we shall omit the index h in φ_i^h.

Using the conjugacy conditions (4), when $x=\xi$ we obtain

$$\varphi_n = \varphi_n^{(0)}+\omega_n^{(0)}+O(h), \qquad \varphi_{n+1} = \varphi_{n+1}^{(0)}+\omega_{n+1}^{(0)}+O(h)\,, \tag{5}$$

$$\varphi_n^{(0)} = \frac{w}{h}\left[B_n^h\left(\frac{1-\theta}{k_+}+\frac{\theta}{k_-}\right)-A_n^h.\frac{1}{k_-}\right]+\frac{1}{2}B_n^h[(1-\theta)^2u_+''-\theta^2u_-'']+$$
$$+(0{\cdot}5+\theta)A_n^hu_-''-(ku')_-', \tag{6}$$

$$\varphi_{n+1}^{(0)} = \frac{w}{h}\left[B_{n+1}^h.\frac{1}{k_+}-A_{n+1}^h\left(\frac{1-\theta}{k_+}+\frac{\theta}{k_-}\right)\right]-\frac{1}{2}A_{n+1}^h[(1-\theta)^2u_+''-\theta^2u_-'']+$$
$$+(1{\cdot}5-\theta)B_{n+1}^hu_+''-(ku')_+',$$

$$w = k_-u_-' = k_+u_+'\,,$$

$$\omega_n^{(0)} = (F_n^h-f_n)-(D_n^h-q_n)u(\xi),$$

$$\omega_{n+1}^{(0)} = (F_{n+1}^h-f_{n+1})-(D_{n+1}^h-q_{n+1})u(\xi)\,. \tag{7}$$

We see from this that the terms φ_n and φ_{n+1} are of the order of $\frac{1}{h}$, i.e. the difference operator $L_h^{(k,q,f)}$ does not approximate to the differential operator $L^{(k,q,f)}$ at the points $x=x_n$, $x=x_{n+1}$.

If $k(x)\in Q^{(1)}$, $q(x)\in Q^{(0)}$, $f(x)\in Q^{(0)}$ then $\varphi_i=\rho(h)$ for $i\neq n,\ n+1$, $\varphi_n=\varphi_n^0+\omega_n^0+\rho(h)$, $\varphi_{n+1}=\varphi_{n+1}^0+\omega_{n+1}^0+\rho(h)$, where $\rho(h)\to 0$ as $h\to 0$.

2. *Conservative schemes*. Suppose now that $L_h^{(k,q,f)}$ is a conservative scheme, i.e. that $B_i^{(h,k)} = A_{i+1}^{(h,k)}$. In this case formulae (6) take the form

$$\varphi_n^{(0)} = \frac{w}{h}\left[A_{n+1}^h\left(\frac{\theta}{k_-}+\frac{1-\theta}{k_+}\right)-A_n^h.\frac{1}{k_-}\right]+\frac{1}{2}A_{n+1}^h[(1-\theta)^2u_+''-\theta^2u_-'']+$$
$$+A_n^h(0{\cdot}5+\theta)u_-''-(L^{(k)}u)_-+\theta h(L^{(k)}u)_-', \tag{8}$$

$$\varphi_{n+1}^{(0)} = \frac{w}{h}\left[A_{n+2}^h \cdot \frac{1}{k_+} - A_{n+1}^h\left(\frac{\theta}{k_-} + \frac{1-\theta}{k_+}\right)\right] - \frac{1}{2} A_{n+1}^h[(1-\theta)^2 u_+'' - \theta^2 u_-''] +$$
$$+ A_{n+2}^h(1{\cdot}5-\theta)u_+'' - (L^{(k)}u)_+ - h(1-\theta)\,(L^{(k)}u)_+' \,. \qquad (9)$$

We shall need the sum

$$\varphi_n^{(0)} + \varphi_{n+1}^{(0)} = \frac{w}{h}\left(\frac{1}{k_+} A_{n+2}^h - \frac{1}{k_-} A_n^h\right) + A_n^h(\theta + 0{\cdot}5)u_-'' + A_{n+2}^h(1{\cdot}5-\theta)u_+'' -$$
$$(L^{(k)}u) - (L^{(k)}u)_+ + O(h)\,. \qquad (10)$$

If L_h^k is the initial scheme of 2nd order approximation, then

$$A_n^h = k_- - (0{\cdot}5+\theta)hk_-' + O(h^2)\,, \qquad A_{n+2}^h = k_+ + (1{\cdot}5-\theta)hk_+' + O(h^2)$$

and therefore

$$\frac{1}{k_+} A_{n+2}^h - \frac{1}{k_-} A_n^k = h\left[(1{\cdot}5-\theta)\frac{k_+'}{k_+} + (0{\cdot}5+\theta)\cdot\frac{k_+'}{k_-}\right] + O(h^2)\,.$$

As a result we obtain

$$\varphi_n^{(0)} + \varphi_{n+1}^{(0)} = (0{\cdot}5-\theta)\,\{(L^{(k)}u)_+ - (L^{(k)}u)_-\} + O(h). \qquad (11)$$

Now calculate the sum $\omega_n^0 + \omega_{n+1}^0$. For the sake of simplicity we shall make the calculations relating to the term containing f.

We must distinguish between two cases:

(a) if $\theta < 0{\cdot}5$, then

$$F_{n+1}^h = f_+ + O(h)\,, \qquad F_n^h = F_n^{(0)} + O(h)\,,$$
$$\omega_n^0 + \omega_{n+1}^0 = F_n^{(0)} - f_- + O(h)\,, \qquad F_n^{(0)} = F^{(0)}[f(x_n+sh)]\,,$$

(b) if $\theta > 0{\cdot}5$, then

$$F_{n+1}^h = F_{n+1}^{(0)} + O(h)\,, \qquad F_n^h = f_- + O(h)\,,$$
$$\omega_n^0 + \omega_{n+1}^0 = F_{n+1}^0 f_+ + O(h)\,.$$

In the general case of a scheme $L_h^{(k,q,f)}$ of second order approximation we obtain:

$$\varphi_n + \varphi_{n+1} = (0{\cdot}5-\theta)\,[(L^{(k,q,f)}u)_+ - (L^{(k,q,f)}u)_-] + \chi_n + O(h) = \chi_n + O(h)\,, \qquad (12)$$

where

$$\chi_n = F_n^{(0)} - D_n^{(0)}u(\xi) - (0{\cdot}5+\theta)\,(f_- - q_- u(\xi)) + (0{\cdot}5-\theta)(f_+ - q_+ u(\xi))\,, \qquad \theta < 0{\cdot}5\,,$$
$$\chi_n = F_{n+1}^{(0)} - D_{n+1}^{(0)}u(\xi) + (1{\cdot}5-\theta)\,(f_- - q_- u(\xi)) - (1{\cdot}5-\theta)\,(f_+ - q_+ u(\xi))\,, \qquad \theta > 0{\cdot}5\,.$$

We note that when $q(x) = 0$, $f(x) = 0$

$$\varphi_n + \varphi_{n+1} = O(h)\,. \qquad (13)$$

While in the general case

$$\varphi_n + \varphi_{n+1} = O(1)\,. \qquad (14)$$

3. *A basic lemma.* Since the difference operator does not approximate to the differential operator in the neighbourhood of a point of discontinuity of the coeffi-

cients of the equation $L^{(k,q,f)}u=0$ the question arises as to what necessary conditions a φ_n and φ_{n+1} must satisfy for the scheme $L_h^{(k,q,f)}$ to converge or have nth order accuracy ($n=1, 2$) on any sequence of nets S_h as $h\to 0$ (or $N\to\infty$).

The answer to this question will be obtained with the help of the basic lemma to be proved in this section.

Consider the difference operator

$$\tilde{L}_h z_i=\frac{1}{h^2}(\tilde{B}_i^h\Delta z_i-\tilde{A}_i^h\nabla z_i), \tag{15}$$

defined on any sequence of nets Sh $\left(h=\frac{1}{N}\right)$.

Let $(\bar{x}, \bar{\bar{x}})$ be some neighbourhood of the particular point $\xi\in(0,1)$, $0<\bar{x}<\xi<\bar{\bar{x}}<1$, where

$$\bar{x}=x_{\bar{n}-1}+\bar{\theta}h,\quad \xi=x_n+\theta h,\quad \bar{\bar{x}}=x_{\bar{\bar{n}}}+\bar{\bar{\theta}}h,\quad 0\leqslant\bar{\theta},\theta,\bar{\bar{\theta}}<1,$$

and

$$x_i=ih.$$

We shall be concerned with the operators $\tilde{L}_h$ whose coefficients satisfy the conditions

$$0<M_1\leqslant\tilde{A}_i^h\leqslant M_2,\quad 0<M_1\leqslant\tilde{B}_i^h\leqslant M_2,\quad \bar{n}<i<\bar{\bar{n}}, \tag{$\bar{\alpha}$}$$

$$e^{-bh}\leqslant\tilde{\varkappa}_i=\frac{\tilde{B}_i^h}{\tilde{A}_{i+1}^h}\leqslant e^{bh},\quad \bar{n}<i<\bar{\bar{n}},\quad i\neq n-1, n, n+1, \tag{$\bar{\beta}$}$$

where M_1, M_2 and b are positive constants, independent of h.

BASIC LEMMA. *If the coefficients $\tilde{A}_i^h$ and $\tilde{B}_i^h$ of the operator $\tilde{L}_h$ satisfy the conditions $(\bar{\alpha})$ and $(\bar{\beta})$ and the function $\tilde{\varphi}_i^h$ converges uniformly to zero on the interval $(\bar{x}, \bar{\bar{x}})$ as $h\to 0$ for all $i\neq n, n+1$:*

$$|\tilde{\varphi}_i^h|\leqslant\rho(h), \tag{16}$$

then the necessary conditions that some sequence of solutions of the equation

$$\tilde{L}_h z_i^h=-\tilde{\varphi}_i^h \tag{17}$$

converge uniformly to zero as $h\to 0$ on the interval $(\bar{x}, \bar{\bar{x}})$ ($|z_i^h|\leqslant\rho(h)$, $\bar{n}\leqslant i\leqslant\bar{\bar{n}}$) are

$$h^2\tilde{\varphi}_{n+1}^h=\rho(h),\quad h^2\tilde{\varphi}_n^h=\rho(h), \tag{a}$$

$$\Delta(\xi, h)=h(\tilde{A}_{n+1}^h\tilde{\varphi}_n^h+\tilde{B}_n^h\tilde{\varphi}_{n+1}^h)=\rho(h). \tag{b}$$

We introduce the functions

$$\varphi_i^{(1)}=\delta_{in}\tilde{\varphi}_n^h+\delta_{i,\,n+1}\tilde{\varphi}_{n+1}^h,\quad \varphi_i^{(2)}=\tilde{\varphi}_i^h-\varphi_i^{(1)},$$

where $\delta_{ij}=\begin{cases}1, & i=j,\\ 0, & i\neq j,\end{cases}$ and put the considered solution z_i^h of equation (17) in the form of the sum

$$z_i^h=z_i^{(1)}+z_i^{(2)},$$

where $z_i^{(2)}$ is a solution of the equation $L_h z_i^{(2)} = -\varphi_i^{(2)}$ with some boundary conditions, such as $z_{\bar{n}}^{(2)} = 0$, $z_{\bar{\bar{n}}}^{(2)} = 0$.

As we showed in § 2, the conditions $(\bar{\alpha})$ and $(\bar{\beta})$ guarantee the existence and two-sided boundedness of Green's difference function of the operator $\tilde{L}_h$.

In proving the basic lemma we introduce Green's difference functions $G_{ij}^{(a)}$ and $G_{ij}^{(b)}$ of the conservative operator $\Lambda_i \tilde{L}_h$ for the first boundary problem on the segments $n \leqslant i \leqslant \bar{\bar{n}}$ and $\bar{n} \leqslant i \leqslant \bar{\bar{n}}$ respectively.

The functions $G_{ij}^{(a)}$ and $G_{ij}^{(b)}$ are determined from the conditions

$$\Lambda_i^{(a)} \tilde{L}_h G_{ij}^{(a)} = -\frac{\delta_{ij}}{h} \qquad (n < i < \bar{\bar{n}};\ \ j \text{ fixed},\ \ n < j \leqslant \bar{\bar{n}}),$$

$$G_{ij}^{(a)} = 0 \quad \text{when } i = n \text{ and } i = \bar{\bar{n}}, \qquad \Lambda_i^{(a)} = \sum_{s=n}^{i-1} \tilde{\varkappa}_s;$$

$$\Lambda_i^{(b)} L_h G_{ij}^{(b)} = -\frac{\delta_{ij}}{h} \qquad (\bar{n} < i < \bar{\bar{n}};\ \ j \text{ fixed},\ \bar{n} \leqslant j \leqslant \bar{\bar{n}}),$$

$$G_{ij}^{(b)} = 0 \quad \text{when } i = \bar{n} \text{ and } i = \bar{\bar{n}}, \quad \Lambda_i^{(b)} = \prod_{s=n}^{i-1} \tilde{\varkappa}_s .$$

Using Green's formula, we find

$$z_i^{(2)} = \sum_{i=\bar{n}+1}^{\bar{\bar{n}}} G_{ij}^{(b)} \Lambda_i^{(b)} \varphi_j^{(2)} h \qquad (z_{\bar{n}}^{(2)} = z_{\bar{\bar{n}}}^{(2)} = 0).$$

From this and that fact that $G_{ij}^{(b)}$ is bounded if follows that

$$\| z_i^{(2)} \|_1 = \rho(h) \qquad (\| \psi \|_1 = \max_{\bar{n} \leqslant i \leqslant \bar{\bar{n}}} | \psi_i |). \tag{18}$$

Thus on the segment $\bar{n} \leqslant i \leqslant \bar{\bar{n}}$ instead of considering the solution of equation (17) we can consider the solution $z_i^{(1)}$ of the equation

$$L_h z_i^{(1)} = -\varphi_i^{(1)}. \tag{19}$$

By hypothesis $\| z_i^h \|_1 = \rho(h)$. From this and (18) it follows that $z_i^{(1)} = \rho(h)$ on $(\bar{x}, \bar{\bar{x}})$.

It is required to prove that given this condition, conditions (a) and (b) must be satisfied.

To prove the Lemma, we put the solution of equation (19) in terms of the Green's functions $G_{ij}^{(a)}$ and $G_{ij}^{(b)}$. Thus, for example, the solution on the interval $n < i < \bar{\bar{n}}$ is expressed in terms of $\tilde{\varphi}_{n+1}^h$ and the boundary values $z_n^{(1)}$, $z_{\bar{\bar{n}}}^{(1)}$, tending to zero as $h \to 0$. To obtain the necessary estimates (a) or (b) we use the lower estimates for Green's difference function (see § 2, Section 3).

(1) We put the solution of (19) inside the interval $n < i \leqslant \bar{\bar{n}}$ in the form

$$z_i^{(1)} = \bar{z}_i^{(1)} + \bar{\bar{z}}_i^{(1)}, \qquad \bar{z}_i^{(1)} = \sum_{i=n+1}^{\bar{\bar{n}}-1} G_{ij}^{(a)} \Lambda_j^{(a)} \tilde{\varphi}_j^h h = G_{i,n+1}^{(a)} \Lambda_{n+1}^{(a)} \tilde{\varphi}_{n+1}^h h, \tag{20}$$

$$\bar{\bar{z}}_i^{(1)} = \frac{1}{h} \tilde{A}_{n+1}^h G_{i,n+1}^{(a)} \Lambda_{n+1}^{(a)} z_n^{(1)} + \frac{1}{h} \tilde{B}_{\bar{\bar{n}}-1}^h G_{i,\bar{\bar{n}}-1}^{(a)} \Lambda_{\bar{\bar{n}}-1}^{(a)} \cdot z_{\bar{\bar{n}}}^{(1)}. \tag{21}$$

Due to the fact that the difference rations of Green's functions $G_{ij}^{(a)}$ is bounded for $j = n$ and $j = \bar{\bar{n}} - 1$ we have

$$\| \bar{\bar{z}}_i^{(1)} \|_1 \leq M\left(|z_n^{(1)}| + |z_{\bar{\bar{n}}-1}^{(1)}|\right) = \rho(h),$$

since $z_i^{(1)} = \rho(h)$.

Using the lower estimate for $G_{ij}^{(a)}$ (see § 2, Section 3, (22)), we see that

$$\Lambda_{n+1}^{(a)} G_{i,n+1}^{(a)} \geq \frac{M_1'}{(M_2')^2} \psi_{i,n+1} = \frac{M_1'}{(M_2')^2} h(x_{\bar{\bar{n}}} - x_i)\,.$$

For definiteness choose $i = i_0$, such that $x_{\bar{\bar{n}}} - x_{i_0} \geq 0{\cdot}5(\bar{\bar{x}} - \xi)$; then

$$\Lambda_{n+1}^{(a)} G_{i_0, n+1}^{(a)} \geq h \cdot g_0^{(a)}, \qquad g_0^{(a)} = \frac{M_1'}{(M_2')^2} \cdot 0{\cdot}5(\bar{\bar{x}} - \xi)\,.$$

From this and (21) it follows that

$$|\tilde{\varphi}_{n+1}^h| \cdot h^2 \leq \max_{n < i < \bar{\bar{n}}} |z_i^{(1)} - \bar{\bar{z}}_i^{(1)}|\,,$$

$$h^2 \tilde{\varphi}_{n+1}^h = \rho(h)\,. \tag{22}$$

(2) To obtain condition (b) we put the solution of equation (19) inside the interval $\bar{n} < i < \bar{\bar{n}}$ in terms of Green's function $G_{ij}^{(b)}$:

$$z_j^{(1)} = \bar{z}_i^{(1)} + \bar{\bar{z}}_i^{(1)},$$

where

$$\bar{z}_i^{(1)} = G_{i,n}^{(b)} \Lambda_n^{(b)} \tilde{\varphi}_n^h . h + G_{i,n+1}^{(b)} \Lambda_{n+1}^{(b)} \tilde{\varphi}_{n+1}^h . h\,,$$

$\bar{\bar{z}}_i^{(1)}$ is expressed by a formula similar to (21).

Remembering that

$$\Lambda_{n+1}^{(b)} = \Lambda_n^{(b)} . (\tilde{B}_n^h / \tilde{A}_{n+1}^h),$$

we put $\bar{z}_i^{(1)}$ in the form

$$\bar{z}_i^{(1)} = G_{i,n}^{(1)} . \frac{\Delta(\xi, h)}{\tilde{A}_{n+1}^h} . \Lambda_n^{(b)} + \frac{G_{i,n+1}^{(b)} - G_{i,n}^{(b)}}{h} . \Lambda_{n+1}^{(b)} (h^2 \tilde{\varphi}_{n+1}^h)\,. \tag{23}$$

Due to (22) the second term is the quantity $\rho(h)$.

We then choose $i = i_0$ such that $x_{i_0} - x_{\bar{n}} \geq 0{\cdot}5(\bar{\bar{x}} - \bar{x})$, and obtain

$$\frac{\Lambda_n^{(b)}}{\tilde{A}_{n+1}^h} G_{i,n}^{(b)} \geq g_0^{(b)}.$$

From (23) we have the estimate

$$\Delta(\xi, h) < \frac{|\bar{z}_{i_0}^{(1)}|}{g_0^{(b)}} + \rho(h) = \rho(h)\,. \tag{24}$$

The condition $h^2 \tilde{\varphi}_n^h = \rho(h)$, clearly, is a consequence of conditions (22) and (24). This proves the lemma.

NOTE. The basic lemma remains valid if the function $\varphi_i^{(2)} = \varphi_i^h - (\delta_{i,n}\tilde{\varphi}_n^h + \delta_{i,n+1}\varphi_{n+1}^h)$ satisfies the convergence condition according to one of the norms we have used:

$$\|\varphi_i^{(2)}\|_1 = \max_{\bar{n}<i<\bar{\bar{n}}} |\varphi_i^{(2)}| = \rho(h),$$

$$\|\varphi_i^{(2)}\|_2 = \sum_{i=\bar{n}+1}^{\bar{\bar{n}}-1} h \left| \sum_{s=\bar{n}+1}^{i} h\varphi_s^{(2)} \right| = \sum_{i=\bar{n}+1}^{\bar{\bar{n}}-1} h \left| \sum_{\substack{s=\bar{n}+1 \\ (s\neq n,\, s\neq n+1)}}^{i} h\varphi_s \right| = \rho(h),$$

$$\|\varphi_i^{(2)}\|_3 = \sum_{i=\bar{n}+1}^{\bar{\bar{n}}-1} h|\varphi_i^{(2)}| = \rho(h).$$

For, from the inequalities

$$\|z_i^{(2)}\|_1 \leqslant M\|\varphi_i^{(2)}\|_\sigma \qquad (\sigma = 1, 2, 3),$$

obtained in § 2, Section 3, it follows that

$$\|z_i^{(2)}\|_1 = \rho(h).$$

LEMMA 1. *If the coefficients of the operator $\tilde{L}_h$ satisfy the conditions $(\bar{\alpha})$ and $(\bar{\beta})$, and the function $\varphi_i^{(2)} = \tilde{\varphi}_i^{(h)} - (\delta_{i,n}\tilde{\varphi}_n^h + \delta_{i,n+1}\tilde{\varphi}_{n+1}^h)$ on the interval $(\bar{x}, \bar{\bar{x}})$ is of the order of h^m according to some norm $\|\varphi_i^{(2)}\|_\sigma$ $(\sigma = 1, 2, 3)$:*

$$\|\varphi_i^{(2)}\|_\sigma = O(h^m),$$

then the necessary conditions for some sequence of solutions of the equation (17) *to have* mth *order smallness as* $h\to 0$ $(\|\tilde{z}_i^h\|_1 = O(h^m))$ *on the interval* $(\bar{x}, \bar{\bar{x}})$, $(m = 1, 2)$, *are*

$$h^2\tilde{\varphi}_n^h = O(h^m), \quad h^2\varphi_{n+1}^h = O(h^m), \tag{a_m}$$

$$\Delta(\xi, h) = O(h^m). \tag{b_m}$$

4. *The necessary condition for convergence of the initial scheme in the class of discontinuous coefficients.* To examin the necessary conditions for convergence of the difference scheme $L_h^{(k,q,f)}$ from the initial family $\mathcal{L}(n_1, n_2, n_3)$ in the class of discontinuous coefficients, we consider the difference problem (III) for the error $z_i = y_i - u(x_i)$.

We first consider the first rank scheme $L_h^{(k)}$ satisfying (A.C.1).

Let $\xi = x_n + \theta . h$ $(0 < \theta \leqslant 1)$ be a point of discontinuity of the coefficient $k(x) \in Q^{(1)}$. Since, when $i \neq n$, $i \neq n+1$

$$B_i^{(h,k)} = k_i + hk_i' B_1^{(0)}[s] + h\rho(h).$$

$$A_i^{(h,k)} = k_i + hk_i'(1 + A_1^{(0)}[s]) + h\rho(h),$$

we find that $\varkappa_i = \dfrac{B_i^{(h,k)}}{A_{i+1}^{(h,k)}} = 1 + h\rho(h)$ when $i \neq n$ in some neighbourhood $(\bar{x}, \bar{\bar{x}})$ of the point ξ not containing other points of discontinuity of the function $k(x)$.

It follows from this and from the condition $0 \leqslant M_1 \leqslant k(x) \leqslant M_2$ that conditions (α) and (β) are satisfied and

$$\varphi_i^h = (L_h^{(k)}u_i) - (L^{(k)}u)_i = \rho(h)$$

at the points $i \neq n$, $i \neq n+1$ in the interval $(\bar{x} + h, \bar{\bar{x}} - h)$.

Remembering the note at the end of Section 3, we see that the condition

$$\Delta = h(B_n^{(h,k)}\varphi_{n+1}^h + A_{n+1}^{(h,k)}\varphi_n^h) = \rho(h) \tag{23'}$$

is necessary for the convergence to zero of the solution of problem (III), i.e. for the convergence of the difference scheme $L_h^{(k)}$ and, therefore, of the scheme $L_h^{(k,q,f)}$ of $\mathcal{L}(1, 0, 0)$ in the class of discontinuous coefficients.

The condition $h^2\varphi_n^h = \rho(h)$ is satatisfied, since $h\varphi_n^h = O(1)$.

Substituting the expressions (5) and (6) for φ_n^h and φ_{n+1}^h in (23) we obtain the necessary condition for convergence of the scheme $L_h^{(k,q,f)}$ of the family $\mathcal{L}(1, 0, 0)$ in the form

$$\frac{B_n^{(h,k)}B_{n+1}^{(h,k)}}{k_+} - \frac{A_n^{(h,k)}A_{n+1}^{(h,k)}}{k_-} = \rho(h). \tag{24'}$$

Now consider conservative schemes.

THEOREM 1. *Any conservative scheme $L_h^{(k,q,f)}$ of zero rank satisfies the necessary convergence condition in the class $k, q, f \in Q^{(0)}$.*

For, if the scheme is conservative, then $B_n^{(h,k)} = A_{n+1}^{(h,k)}$,

$$A_n^{(h,k)} = A^h[k(x_n + sh)] = A^{(0)}[k(x_n + sh)] + \rho(h) = k_n + \rho(h) = k_- + \rho(h),$$
$$B_{n+1}^{(h,k)} = A_{n+2}^{(h,k)} = k_{n+2} + \rho(h) = k_+ + \rho(h).$$

Putting these expressions in the left-hand side of condition (24′) we shall have

$$\frac{B_n^{(h,k)}B_{n+1}^{(h,k)}}{k_+} - \frac{A_n^{(h,k)}A_{n+1}^{(h,k)}}{k_-} = A_{n+1}^{(h,k)}\left(\frac{B_{n+1}^{(h,k)}}{k_+} - \frac{A_n^{(h,k)}}{k_-}\right) = \rho(h).$$

THEOREM 2. *Any conservative scheme $L_h^{(k,q,f)}$ of first order approximation of the family $\mathcal{L}(2, 1, 1)$ satisfies the necessary condition for first order accuracy in the class of discontinuous coefficients $k(x) \in Q^{(1)}$, $q(x) \in Q^{(0)}$, $f(x) \in Q^{(0)}$.*

Thus the condition $h^2\varphi_n = O(h)$ or $h\varphi_n = O(1)$ is satisfied automatically. As for the second condition $\Delta = O(h)$, it is easily verified, since $B_n^{(h,k)} = A_{n+1}^{(h,k)}$, $B_{n+1}^{(h,k)} = k_+ + O(h)$, $A_n = k_- + O(h)$. From the conditions of the theorem it follows that $\varphi_i = O(h)$ when $i \neq n$, $i \neq n+1$. Therefore Lemma 1 ($m = 1$) can be used here.

5. *The necessary conditions for second order accuracy in the class of discontinuous coefficients.* We consider the difference boundary problem (III) for the function $z_i = y_i - u(x_i)$.

For simplicity we assume that k, q, f have one point of discontinuity. Put $\varphi_i^{(1)} = \delta_{in}\varphi_n + \delta_{i,n+1}\varphi_{n+1}$, $\varphi_i^{(2)} = \varphi_i - \varphi_i^{(1)}$. If $k(x) \in Q^{(3)}$, $q(x) \in Q^{(2)}$, $f(x) \in Q^{(2)}$ and the scheme $L_h^{(k,q,f)}$ belongs to the family $\mathcal{L}(3, 2, 2)$ and has second order approximation, then $\|\varphi_i^{(2)}\|_1 = O(h^2)$ and the conditions of Lemma 1 are satisfied.

From Lemma 1 the necessary conditions for second order accuracy of the scheme $L_h^{(k,q,f)}$ in the class of discontinuous coefficients take the form

$$\varphi_n = O(1) \quad \text{or} \quad \varphi_{n+1} = O(1), \tag{a$_2$}$$

$$\Delta = O(h^2). \tag{b$_2$}$$

If the scheme is conservative, then condition (b_2) is equivalent to condition (b_2'): $\varphi_n + \varphi_{n+1} = O(h)$.

LEMMA 2. *The necessary condition for second order accuracy*

$$\varphi_n + \varphi_{n+1} = O(h) \tag{25}$$

in the class of discontinuous coefficients $k(x) \in Q^{(2)}$ *is satisfied by any conservative scheme* $L_h^{(k)}$ *of second order approximation.*

This is a consequence of the formula

$$\varphi_n^{(0)} + \varphi_{n+1}^{(0)} = (0{\cdot}5 - \theta)\{(L^{(k)}u)_+ - (L^{(k)}u)_-\} + O(h) = O(h),$$

since $(L^{(k)}u)_+ = (L^{(k)}u)_- = 0$, $u = u(x)$ being a solution of the equation $L^{(k)}u = 0$.

From this it follows that it is only necessary to verify the first of the necessary conditions $\varphi_n = O(1)$ for a conservative scheme of second order.

6. *Sufficient conditions for convergence.* In Section 5 we established the necessary conditions for the convergence of the initial scheme $L_h^{(k,q,f)}$ in some class of discontinuous coefficients ($k(x) \in Q^{(1)}$, $q(x) \in Q^{(0)}$ $f(x) \in Q^{(0)}$).

We shall show that this condition is also a sufficient condition for convergence.

THEOREM 3. *The necessary and sufficient condition for the difference scheme* $L_h^{(k,q,f)}$ *of first order approximation from the class* $\mathfrak{L}$ (2, 1, 1) *to converge for any coefficients* $k(x) \in Q^{(1)}$, $q(x) \in Q^{(0)}$, $f(x) \in Q^{(0)}$ *is that at each point* $\xi_j = x_{n_j} + \theta_j h$ $(j = 1, 2, \ldots, j_0)$ *of discontinuity of the functions* k, q *and* f *the condition*

$$\Delta_j = h(B_{n_j}^{(h,k)}\varphi_{n_j+1} + A_{n_j+1}^{(h,k)}\varphi_{n_j}) = \rho(h), \tag{26}$$

is satisfied, where $\varphi_i = L_h^{(k,q,f)}u_i - (L^{(k,q,f)}u)_i$.

Proof. The necessity of the condition $\Delta_j = \rho(h)$ has already been proved. To prove that it is also sufficient we use the formula

$$\| z_i \|_1 = \| y_i - u_i \|_1 \leqslant M \| \varphi_i \|_{\sim} \qquad (\sim = 1, 2, 3),$$

where M is a constant, independent of h.

We put φ_i and z_i in the form of the sums

$$\varphi_i = \varphi_i^{(1)} + \varphi_i^{(2)}, \qquad \varphi_i^{(1)} = \sum_{j=1}^{j_0} (\delta_{i,n_j}\varphi_{n_j} + \delta_{i,n_j+1}\varphi_{n_j+1}), \qquad z_i = z_i^{(1)} + z_i^{(2)},$$

where $z_i^{(m)}$ is a solution of problem (III) with the right-hand side equal to $\varphi_i^{(m)}$ $(m = 1, 2)$. At the points $i \neq n_j$, $i \neq n_j + 1$ $(j = 1, 2, \ldots, j_0)$, $\varphi_i^{(2)} = \rho h$, using the inequality

$$\| z_i^{(2)} \|_1 \leqslant \frac{1}{M_1'} \| \varphi_i^{(2)} \|_1,$$

we obtain $\| z_i^{(2)} \|_1 = \max\limits_{0<i<N} | z_i^{(2)} | = \rho(h)$.

For an estimate of $z_i^{(1)}$ we use the inequality

$$\| z_i^{(1)} \|_1 \leqslant M \| \Lambda_s \varphi_s^{(1)} \|_2 \qquad \left(\| \psi_s \|_2 = \sum_{i=1}^{N-1} h \left| \sum_{s=1}^{i} h \psi_s \right| \right), \tag{27}$$

where $A_s = \prod_{m=1}^{s-1} (B_m^h / A_{m+1}^h)$ and M is a constant independent of h.

For definiteness we shall take $n_j < n_{j+1}$.

Putting the expression for $\varphi_s^{(1)}$ in (27) we obtain

$$\| z_i^{(1)} \|_1 \leqslant M \sum_{j=1}^{j_0} [\Lambda_{n_j} \varphi_{n_j} h^2 + h(\Lambda_{n_j} \varphi_{n_j} + \Lambda_{n_j+1} \varphi_{n_j+1})] .$$

Putting $\Lambda_{n_j+1} = \Lambda_{n_j}(B_{n_j+1})$ in the first term we find

$$\| z_i^{(1)} \|_1 \leqslant M \left\{ \sum_{j=1}^{j_0} \Lambda_{n_j} \varphi_{n_j} h^2 + \sum_{j=1}^{j_0} \frac{\Lambda_{n_j}}{A_{n_j+1}} \Delta_j \right\} . \tag{28}$$

The first term is of the first order of smallness, since $h\varphi_{n_j} = O(1)$. The second term converges to zero as $h \to 0$, since, by hypothesis, $\Delta_j = \rho(h)$ and the number j_0 is finite. Thus

$$\| z_i \|_1 \leqslant \| z_i^{(1)} \|_1 + \| z_i^{(2)} \|_1 = \rho(h) .$$

THEOREM 4. *If $L_h^{(k,q,f)}$ is a difference scheme of the type $\mathcal{L}(m+1, m, m)$ of mth order approximation $(m = 1, 2)$, then the necessary and sufficient condition for it to have mth order accuracy in the class $k(x) \in Q^{(m+1)}$, $q, f \in Q^{(m)}$ is that in the neighbourhood of each point of discontinuity $\xi_j = x_{n_j} + \theta_j h$ $(j = 1, 2, \ldots, j_0)$ of the functions k, q, f the conditions*

$$h^2 \varphi_{n_j}^h = O(h^m), \qquad h^2 \varphi_{n_j+1}^h = O(h^m), \tag{a$_m$}$$

$$\Delta(\xi_j, h) = O(h^m) \qquad (j = 1, 2, \ldots, j_0). \tag{b$_m$}$$

hold.

The sufficiency of conditions (a_m) and (b_m) is proved by the relations $\varphi_i^{(2)} = O(h^m)$, $\| z_i^{(2)} \|_1 = O(h^m)$, and the inequality (28). The necessity follows from Lemma 1.

§ 4. COEFFICIENT-STABLE DIFFERENCE SCHEMES

When solving the differential equation $L^{(k,q,f)}$ by the method of finite differences we sometimes find that the information we have about the coefficients of the equation, k, q, f, is insufficiently complete. This can happen, for example, when coefficients are determined approximately using some computing algorithm.

For this reason, even when the coefficients of the difference scheme $L_h^{(k,q,f)}$ are calculated exactly, some error may occur. On the other hand, it can happen that the functionals A^h, B^h, D^h, F^h are themselves approximate, and this too leads to an error in the coefficients of the scheme. It is therefore clear how important it is to consider schemes with disturbed coefficients.

We introduce below the norm of the disturbance of the scheme coefficients and with its help we give a definition of a coefficient-stable, or co-stable, difference scheme.

The principal result of this paragraph is the theorem which states that the necessary and sufficient condition for the co-stability of a canonical scheme of type $\mathcal{C}(1,0,0)$ is that it shall be conservative (Theorem 3).

1. *The dependence of the solution of a differential equation on its coefficients.* Consider the boundary problem

$$L^{(k,q,f)}u=0,\quad 0<x<1,\quad u(0)=\bar{u}_1,\quad u(1)=\bar{u}_2 \tag{1}$$

and compare its solution $u(x)$ with the solution $\tilde{u}(x)$ of the disturbed problem

$$L_{\tilde{u}}^{(\tilde{k},\tilde{q},\tilde{f})}=0,\quad 0<x<1,\quad \tilde{u}(0)=\bar{u}_1,\quad \tilde{u}(1)=\bar{u}_2. \tag{2}$$

We shall assume that the coefficients of problems (1) and (2) satisfy the conditions

$$0<M_1\leqslant k(x)\leqslant M_2,\quad 0\leqslant q(x)\leqslant M_3,\quad |f(x)|\leqslant M_4. \tag{α}$$

LEMMA 1. *If the coefficients of equations* (1) *and* (2) *are piece-wise continuous and satisfy conditions* (α), *then*

$$|u(x)-\tilde{u}(x)|\leqslant C_1\int_0^1|k(x)-\tilde{k}(x)|\,dx+C_2\int_0^1|q(x)-\tilde{q}(x)|\,dx+C_3\int_0^1|f(x)-\tilde{f}(x)|\,dx, \tag{3}$$

where C_1, C_2, C_3 are positive constants depending only on M_j ($j=1,2,3,4$), $\bar{u}_1$ and $\bar{u}_2$.

For, the difference $z=u-\tilde{u}$ is found from

$$L^{(k,q)}z=-\varphi,\quad z(0)=0,\quad z(1)=0, \tag{4}$$

where

$$\varphi=L^{(k,q,f)}\tilde{u}-L^{(\tilde{k},\tilde{q},\tilde{f})}\tilde{u}=[(k-\tilde{k})\tilde{u}']'-(q-\tilde{q})\tilde{u}+(f-\tilde{f}). \tag{5}$$

We put the solution of (4) in the form

$$z(x)=\int_0^1 G(x,\xi)\varphi(\xi)\,d\xi=-\int_0^1[k(\xi)-\tilde{k}(\xi)]\frac{dG}{d\xi}(x,\xi)\tilde{u}'(\xi)\,d\xi-$$

$$-\int_0^1 G(x,\xi)[q(\xi)-\tilde{q}(\xi)]\tilde{u}(\xi)\,d\xi+\int_0^1 G(x,\xi)[f(\xi)-\tilde{f}(\xi)]\,d\xi, \tag{6}$$

where $G(x,\xi)$ is Green's function for problem (1).

Using the estimates for Green's function and the solutions of problem (1) found by analogy with the estimates found in § 2:

$$0\leqslant G(x,\xi)\leqslant\frac{1}{M_1},\quad \left|\frac{dG}{d\xi}\right|\leqslant\frac{M_1+M_3}{M_1^2},\quad |u|\leqslant M_5,$$

$$|u'|\leqslant M_5(M_1+M_3)/M_1^2,$$

where

$$M_5 = [M_4+(|\bar{u}_1|+|\bar{u}_2|)M_3]/M_1+|\bar{u}_2|+|\bar{u}_1|,$$

we obtain inequality (3) from (6).

The inequality (3) expresses the stability of the solution of problem (1) with respect to a change in the coefficients of the equation.

2. *The principle of co-stability of difference schemes.* Turning to difference schemes, it is natural to demand that they also shall possess the property of stability with respect to a disturbance of their coefficients, whatever its nature.

Together with the initial scheme $L_h^{(k,q,f)}$ we consider the "disturbed" scheme $\tilde{L}_h^{(\tilde{k},\tilde{q},\tilde{f})}$ whose coefficients $\tilde{A}_i^h$, $\tilde{B}_i^h$, $\tilde{D}_i^h$, $\tilde{F}_i^h$ are obtained from the coefficients A_i^h, B_i^h, D_i^h, F_i^h, after an arbitrary disturbance. In the general case this disturbance can be caused by a distortion of the coefficients of the differential equation $L^{(k,q,f)}u=0$, by a distortion of the functionals $A^h[\psi]$, $B^h[\psi]$, $D^h[\psi]$ and $F^h[\psi]$ and by errors arising in the calculation of the scheme coefficients at points of the net.

As well as problem (II) we shall consider the problem

$$\tilde{L}_h^{(\tilde{k},\tilde{q},\tilde{f})}\tilde{y}_i = 0, \qquad 0 < i < N, \quad _0 = \bar{u}_1, \quad \tilde{y}_N = \bar{u}_2. \tag{$\tilde{\text{II}}$}$$

For an estimate of the magnitude of the distortion in the coefficients we introduce some norm, such as

$$\|\psi_i\|_3 = \sum_{i=1}^{N-1} |\psi_i|\, h. \tag{7}$$

For an estimate of the solution we first use the norm

$$\|z_i\|_1 = \max_{0<i<N} \|z_i\|.$$

We shall say that the scheme $L_h^{(k,q,f)}$ is coefficient-stable, or co-stable, if, when the coefficients of any form of the disturbed scheme $\tilde{L}_h^{(\tilde{k},\tilde{q},\tilde{f})}$ converge as $h \to 0$ to the coefficients of the scheme $L_h^{(k,q,f)}$ according to the norm (7), the solution of problem ($\tilde{\text{II}}$) converges uniformly to the solution of problem (I) with the condition that the coefficients $k(x)$, $q(x)$ and $f(x)$ belong to some class $Q^{(m)}$ $(m \geqslant 0)$.

In other words, the scheme $L_h^{(k,q,f)}$ is co-stable if the conditions

$$\begin{gathered} \|\tilde{A}_i^h - A_i^h\|_3 = \rho(h), \quad \|\tilde{B}_i^h - B_i^h\|_3 = \rho(h), \quad \|\tilde{D}_i^h - D_i^h\|_3 = \rho(h), \\ \|\tilde{F}_i^h - F_i^h\|_3 = \rho(h) \end{gathered} \tag{9}$$

mean that

$$\|\tilde{z}_i\|_1 = \|\tilde{y}_i - u(x_i)\|_1 = \rho(h). \tag{10}$$

We see from the definition that the co-stability of a scheme reduces to two requirements: (1) the convergence of the scheme ($\|y_i^h - u(x_i)\|_1 = \rho(h)$ in $Q^{(m)}$ $(m \geqslant 0)$); (2) the stability of the solution of the difference boundary problem (II) with respect to a disturbance of the coefficients of the scheme ($\|\tilde{y}_i - y_i\|_1 = \rho(h)$):

$$\|\tilde{y}_i - u(x_i)\|_1 \leqslant \|\tilde{y}_i - y_i\|_1 + \|y_i - u(x_i)\|_1. \tag{11}$$

In order to study the structure of co-stable difference schemes belonging to the initial family we specify the type of disturbance of the coefficients of the difference equation, assuming that it is caused by a disturbance of the coefficients of the differential equation. Let $\tilde{k}(x, h)$, $\tilde{q}(x, h)$ and $\tilde{f}(x, h)$ be the disturbed coefficients of the differential equation, depending on the step h of the difference net S_h. The corresponding coefficients of the equation $\tilde{L}_h^{(\tilde{k},\tilde{q},\tilde{f})} y_i = 0$ are equal to

$$\tilde{A}_i^h = A^h[\tilde{k}(x_i + sh, h)], \quad \tilde{B}_i^h = B^h[\tilde{k}(x_i + sh, h)],$$

$$\tilde{D}_i^h = D^h[\tilde{q}(x_i + sh, h)], \quad \tilde{F}_i^h = F^h[\tilde{f}(x_i + sh, h)].$$

In particular, in Section 4, we shall consider disturbances of the coefficients k, q and f on one interval of the net in the neighbourhood of a point of discontinuity of these functions.

3. *The co-stability of a conservative scheme.* We show that conservative schemes are co-stable.

Consider first the solutions y_i^h and $\tilde{y}_i^h$ of the two difference boundary problems:

$$L_h y_i^h = \frac{1}{h^2} \Delta(A_i^h \nabla y_i^h) - D_i^h y_i^h = -F_i^h, \quad 0 < i < N, \ y_0^h = \bar{u}_1, \ y_N^h = \bar{u}_2, \tag{12}$$

$$\tilde{L}_h \tilde{y}_i^h = \frac{1}{h^2} \Delta(\tilde{A}_i^h \nabla \tilde{y}_i^h) - \tilde{D}_i^h \tilde{y}_i^h = -F_i^h, \quad 0 < i < N, \ \tilde{y}_0^h = \bar{u}_1, \ \tilde{y}_N^h = \bar{u}_2, \tag{13}$$

where L_h and $\tilde{L}_h$ are conservative difference operators.

Lemma 2. *If the coefficients of the conservative difference equations* (12) *and* (13) *satisfy the conditions* (α), *then the inequality*

$$\| \tilde{y}_i^h - y_i^h \|_1 \leqslant C_1 \| \tilde{A}_i^h - A_i^h \|_2 + C_2 \| \tilde{D}_i^h - D_i^h \|_2 + C_3 \| \tilde{F}_i^h - F_i^h \|_2 \tag{14}$$

holds, where C_1, C_2 and C_3 are positive constants depending only on the constants M_j ($j = 1, 2, 3, 4$).

To prove this lemma it is sufficient to form an equation for the difference $z_i^h = \tilde{y}_i^h - y_i^h$, then to put z_i^h in terms of Green's difference function for problem (13), and, applying Green's first difference formula, to use the estimates of §2 for Green's function and its difference ratios.

We return now to the question of the co-stability of the conservative scheme $L_h^{(k,q,f)}$. We must compare the solutions of the problems (II) and ($\tilde{\text{II}}$) with the condition that

$$\| \tilde{A}_i^h - A_i^h \|_3 = \rho(h), \qquad \| \tilde{D}_i^h - D_i^h \|_3 = \rho(h), \qquad \| \tilde{F}_i^h - F_i^h \|_3 = \rho(h). \tag{15}$$

Lemma 2 is applicable to a conservative scheme of any rank. Therefore we have

$$\| \tilde{y}_i^h - y_i^h \|_1 = \rho(h). \tag{16}$$

From this and inequality (14) it follows that the proof of the co-stability of a conservative scheme reduces entirely to the proof of the convergence of this scheme in the class of discontinuous coefficients, since

$$\|\tilde{y}^h - u\|_1 \leqslant \|y^h - u\|_1 + \rho(h).$$

From the theorem of § 3, Section 6, it follows that the conservative scheme $L_h^{(k,q,f)}$ of the class $\mathcal{L}(1, 0, 0)$ is convergent if

$$k(x) \in Q^{(1)}, \quad q(x) \in Q^{(0)}, \quad f(x) \in Q^{(0)}.$$

This proves

THEOREM 1. *The conservative difference scheme $L_h^{(k,q,f)}$ of the first rank is co-stable if $k(x) \in Q^{(1)}$, $q, f \in Q^{(0)}$.*

In § 5 we shall prove a theorem concerning the co-stability of a conservative scheme of zero rank in the class $Q^{(0)}$ $(k, q, f \in Q^{(0)})$.

Can we make the reverse assertion, i.e. is every co-stable scheme conservative? An affirmative answer to this question will be given in Section 5.

4. *A necessary condition for co-stability.* In order to be able to make practical use of the co-stability requirement for a scheme, we find a specific necessary condition for co-stability similar to the necessary condition for convergence of § 3, Section 3.

Let $L_h^{(k,q,f)}$ be some co-stable scheme of the type $\mathcal{L}(1, 0, 0)$. Since we are discussing a necessary condition, we can discuss the simplest case of a scheme $L_h^{(k)}$ of the first rank, putting $q(x) \equiv 0$ and $f(x) \equiv 0$.

Let $k(x)$ be some function of $Q^{(m)}$ $(m \geqslant 1)$ having a discontinuity at the irrational point $\xi = x_n + \theta . h$, $x_n = n . h$, $0 < \theta < 1$. We introduce the function $k(x, h)$ which coincides with $k(x)$ everywhere except on the interval (x_n, x_{n+1}). Then the coefficients $\tilde{A}_i^h = A_i^{(h,\tilde{k})}$ and $\tilde{B}_i^h = B_i^{(h,\tilde{k})}$ will coincide with $A_i^h = A^{(h,k)}$ and $B_i^h = B_i^{(h,k)}$ everywhere except for $i = n$ and $i = n+1$.

Since the scheme $L_h^{(k)}$ is co-stable, by hypothesis, (i.e. $\|\tilde{y}^h - u\| = \rho(h)$) and is of first rank, we can apply the basic lemma of § 3. Using the expressions for $\tilde{\varphi}_n$ and $\tilde{\varphi}_{n+1}$ for the scheme $L_h^{(\tilde{k})}$ which were given in Section 1 of § 3 we obtain the condition

$$\frac{\tilde{B}_n^h \tilde{B}_{n+1}^h}{k_+} - \frac{\tilde{A}_n^h \tilde{A}_{n+1}^h}{k_-} = \rho(h), \tag{17}$$

which is thus the necessary condition for co-stability.

For the sake of simplicity we shall suppose that $k(x)$ is a piece-wise constant function ($k(x) = k_-$ for $x < \xi$ and $k(x) = k_+$ for $x \geqslant \xi$).

For the function $\tilde{k}(x, h)$ we introduce the arbitrary piece-wise continuous positive function $\mu^*(s)$, $0 \leqslant s \leqslant 1$ and also the function

$$\mu(s) = \begin{cases} k_-, & s \leqslant 0, \\ \mu^*(s), & 0 < s < 1, \\ k_+, & s \geqslant 1 \end{cases} \tag{18}$$

and put

$$\tilde{k}(x, h) = \mu\left(\frac{x - x_n}{h}\right). \tag{19}$$

Since the coefficient $\tilde{k}(x, h)$ must satisfy the lower bound condition ($\tilde{k}(x, u) \geqslant M_1 > 0$) the function $\mu(s)$ must also be subject to this condition.

We shall call functions of the same structure as $\mu(s)$ functions of type μ.

Putting (19) in condition (17), we obtain the necessary condition for co-stability

$$\frac{B[\mu(s)]B[\mu(1+s)]}{k_+} - \frac{A[\mu(s)]A[\mu(1+s)]}{k_-} = 0, \tag{20}$$

where $A[\mu]$ and $B[\mu]$ are the canonical parts of the functionals $A^h[\mu]$ and $B^h[\mu]$ (we have omitted the index (0) from the functionals $A^{(0)}$ and $B^{(0)}$).

We now require that condition (20) shall be satisfied for the function $\mu(s) + \delta.\varphi(s)$ (of type μ), where δ is an arbitrary non-negative parameter,

$$\varphi(s) = \begin{cases} k_-, & s \leqslant 0, \\ \varphi^*(s), & 0 < s < 1, \\ 0, & s \geqslant 1, \end{cases} \tag{21}$$

where $\varphi^*(s)$ is an arbitrary piece-wise continuous non-negative function. Then we shall have

$$\frac{B[\mu(s)+\delta.\varphi(s)]B[\mu(1+s)+\delta.\varphi(1+s)]}{k_+} - \frac{A[\mu(s)+\delta.\varphi(s)]A[\mu(1+s)+\delta.\varphi(1+s)]}{k_-(1+\delta)} = 0. \tag{22}$$

Then using the expansions

$$A[\mu + \delta.\varphi] = A[\mu] + \delta.A_1[\mu, \varphi] + \delta.\rho(\delta),$$
$$B[\mu + \delta.\varphi] = B[\mu] + \delta.B_1[\mu, \varphi] + \delta.\rho(\delta),$$

we obtain from (22), because of the fact that δ is arbitrary,

$$1 + \beta[\mu(s), \varphi(s)] + \beta[\mu(1+s), \varphi(1+s)] = \alpha[\mu(s), \varphi(s)] + \alpha[\mu(1+s), \varphi(1+s)], \tag{23}$$

where $\beta[f, \varphi] = \dfrac{B_1[f, \varphi]}{B[\varphi]}$, $\alpha[f, \varphi] = \dfrac{A_1[f, \varphi]}{A[f]}$ are the logarithmic derivatives of the functionals $B[f]$ and $A[f]$.

Similarly we find

$$1 + \alpha[\mu(s), \psi(-1+s)] + \alpha[\mu(1+s), \varphi(s)] = \beta[\mu(s), \psi(-1+s)] + \beta[\mu(1+s), \psi(s)], \tag{24}$$

where

$$\psi(s) = \begin{cases} 0, & s \leqslant -1, \\ \psi^*(s), & -1 < s < 0, \\ k_+, & s \geqslant 0, \end{cases}$$

and $\psi^*(s)$ is an arbitrary non-negative piece-wise continuous function.

The identities (23) and (24) are also the necessary conditions for the co-stability of the scheme $L_h^{(k)}$ and are used in Section 5.

5. *The conservativeness of a co-stable canonical scheme.* We now require the canonical scheme $L_h^{(k)}$ of the first rank to satisfy the necessary condition for co-stability and, therefore, relations (23) and (24).

LEMMA 3. *If condition* (23) *is satisfied, then*

$$\beta[\mu(s),\ \omega(s)] = 0, \tag{25}$$

where $\omega(s)$ *is an arbitrary non-negative piece-wise continuous function*

$$\omega(s) \not\equiv 0,\ -1 \leqslant s < 0, \qquad \omega(s) \equiv 0 \text{ when } s \geqslant 0.$$

We take the step function

$$\omega_0(s) = \begin{cases} k_-, & s < 0, \\ 0, & s \geqslant 0. \end{cases} \tag{26}$$

Since $\omega_0(1+s) = 0,\ -1 \leqslant s \leqslant 1$ we find

$$\alpha[\mu(1+s),\ \omega_0(1+s)] = \beta[\mu(1+s),\ \omega_0(1+s)] = 0.$$

When $\varphi(s) = \omega_0(s)$ condition (23) becomes

$$1 + \beta[\mu(s),\omega_0(s)] = \alpha[\mu(s),\omega_0(s)]. \tag{27}$$

The pattern functionals $A[f]$ and $B[f]$ of the scheme $L_h^{(k)}$ of any rank are normalized and non-increasing functionals. It follows that the functionals $\alpha[f,\varphi]$ and $\beta[f,\varphi]$ are positive with respect to φ and

$$\beta[f,f] = 1, \quad 0 \leqslant \beta[f,\varphi] \leqslant 1, \quad 0 \leqslant \alpha[f,\varphi] \leqslant 1 \quad \text{for } 0 \leqslant \varphi \leqslant f,\ 0 < \varepsilon \leqslant f$$

(see Lemmas 1, 3, 4 of Section 3, § 1), and in particular

$$\beta[\mu(s),\omega_0(s)] \geqslant 0, \quad 0 \leqslant \alpha[\mu(s),\omega_0(s)] \leqslant 1.$$

Therefore (27) is only valid when

$$\beta[\mu(s),\omega_0(s)] = 0, \tag{28}$$

$$\alpha[\mu(s),\omega_0(s)] = 1. \tag{29}$$

Without loss of generality we can take $\omega(s) \leqslant \omega_0(s)$. Since the functional $\beta[f_1,f_2]$ is non-negative with respect to its second argument we have

$$0 \leqslant \beta[\mu(s),\omega(s)] \leqslant \beta[\mu(s),\omega_0(s)] = 0,$$

and, therefore $\beta[\mu(s),\omega(s)] = 0$.

We prove similarly:

LEMMA 3*. *If condition* (24) *is satisfied, then*

$$\alpha[\mu(s), \chi(s)] = 0\,, \tag{30}$$

where $\chi(s) \not\equiv 0$ *only on the interval* $0 < s \leqslant 1$, *and elsewhere is an arbitrary piece-wise continuous non-negative function.*

LEMMA 4. *If* $B[f(s)]$ *is a canonical functional of the first rank, defined for* $f(s) \in Q^{(0)}[-1, 1]$ *and its first differential* $B_1[f, \varphi] = B[f, \varphi].B[f]$ *satisfies condition,* (25), *then* $B[f(s)]$ *does not depend on* $f(s)$ *for* $-1 \leqslant s < 0$.

We first prove the lemma for functions of type μ. Let $\mu_0(s)$ and $\mu_1(s) = \mu_0(s) + \omega_0(s)$ be two functions of type μ which coincide for $s \geqslant 0$ and differ when $s < 0$. We introduce the function $\mu_1(s) = \mu_0(s) + \lambda\omega_0(s)$ where λ is an arbitrary number on the segment $[0, 1]$. If we assume that $B[\mu_0(s)] \neq B[\mu_1(s)]$ then we can find a number $\lambda = \bar{\lambda}$ such that $\frac{\partial}{\partial\lambda} B[\mu_\lambda(s)]|_{\lambda=\bar{\lambda}} = B_1[\mu_{\bar{\lambda}}(s), \omega_0(s)] \neq 0$, and therefore, $B[\mu_\lambda(s), \omega_0(s)] \neq 0$ which contradicts condition (25) of the lemma.

We can put any function $f(s) \in Q^{(0)}[-1, 1]$ (bounded below by the constant $\mu_1 > 0$) in the form

$$f(s) = \mu(s) + \omega(s)\,, \tag{31}$$

where $\mu(s)$ is an arbitrary function of the form (18), and $\omega(s)$ is an arbitrary piece-wise continuous function different from zero only when $s < 0$.

The lemma will be proved if we show that

$$B[f(s)] = B[\mu(s)]\,. \tag{32}$$

Let $\omega_0^{(1)}(s)$ and $\omega_0^{(2)}(s)$ be step functions of the form (26) which are the lower and upper bounds of $\omega(s)$:

$$0 \leqslant \omega_0^{(1)}(s) \leqslant \omega(s) \leqslant \omega_0^{(2)}(s)\,.$$

Then

$$f_1(s) \leqslant f(s) \leqslant f_2(s)\,, \tag{33}$$

where $f_1(s) = \mu(s) + \omega_0^{(1)}(s)$, $f_2(s) = \mu(s) + \omega_0^{(2)}(s)$ are functions of type μ for which the lemma has been proved, so that

$$B[f_j(s)] = B[\mu(s)]\,, \quad j = 0, 1\,.$$

Since $B[f]$ is a non-decreasing functional, and from (33), we have (32).

We prove similarly:

LEMMA 4*. *If* $A[f(s)]$ *is a canonical functional of the first rank defined for* $f(s) \in Q^{(0)}[-1, 1]$ *and satisfying condition* (30) *then it does not depend on the values of* $f(s)$ *for* $s > 0$.

Let us return now to condition (23).

Since $\varphi(1+s) = 0$ for $s \geqslant 0$ ($\varphi(s)$ is any function of the form (21)), by using Lemma 4 we obtain

$$\beta[\mu(1+s), \varphi(1+s)] = 0 .$$

It follows from Lemma 4* that

$$\alpha[\mu(s), \varphi(s)] = \alpha[k_-, k_-] = 1,$$

since $\mu(s) = \varphi(s) = k_-$ when $s \leqslant 0$.

As a result formula (23) takes the form

$$\beta[\mu(s), \varphi(s)] = \alpha[\mu(1+s), \varphi(1+s)], \tag{34}$$

where $\mu(s)$ and $\varphi(s)$ are arbitrary functions of type (18) and (21).

We show that (34) is equivalent to the inequality

$$B[\mu(s)] = A[\mu(1+s)]. \tag{35}$$

To do this we form the functional

$$H[\mu(s)] = \ln \frac{B[\mu(s)]}{A[\mu(1+s)]},$$

which is equal to zero from Lemmas 4 and 4* for $\mu^*(s) = \mu_0^*(s) = k_-^{(0)} = \text{const}$, and prove that it is equal to zero for any function $\mu(s)$. Suppose that for some function $\mu_1(s) = \mu_0(s) + \varphi(s)$ ($\varphi(s)$ is a function of the form (21)) $H[\mu_1(s)] = H[\mu_1(s)] - H[\mu_0(s)] \neq 0$. Then there exists a value $\lambda = \bar{\lambda}$ for which

$$\frac{\partial}{\partial\lambda} H[\mu_\lambda(s)]|_{\lambda=\bar{\lambda}} = \beta[\mu_\lambda(s), \varphi(s)] - \alpha[\mu_\lambda(1+s), \varphi(1+s)] \neq 0,$$

where

$$\mu_\lambda(s) = \mu_0(s) + \lambda\varphi(s), \quad 0 \leqslant \lambda \leqslant 1, \quad \varphi(s) = k_- - k_-^{(0)} \neq 0 \quad \text{for} \quad s \leqslant 0 .$$

This contradicts condition (34) and so means that $H[\mu] = 0$ for any function of type μ.

Now using Lemmas 4 and 4* and the expression (31) we conclude that

$$B[f(s)] = A[f(1+s)],$$

where $f(s)$ is any piece-wise continuous positive function given on the segment $-1 \leqslant s \leqslant 1$.

This proves:

THEOREM 2. *If the difference scheme $L_h^{(k)}$ of the first rank satisfies the necessary condition for co-stability, it is conservative.*

For, condition (36) means that

$$B[k(x_i + sh)] = A[k(x_i + (1+s)h)] = A[k(x_{i+1} + sh)], \quad \text{i.e.} \quad B_i = A_{i+1} .$$

Theorems 1 and 2 lead to:

THEOREM 3. *The necessary and sufficient condition for the co-stability of the canonical scheme $L_h^{(k,q,f)}$ of the family $\mathfrak{C}(1, 0, 0)$ is that it shall be conservative.*

§ 5. CONVERGENCE AND ACCURACY

Having explained in § 4 that only schemes with a conservative canonical part are co-stable, in § 5 we shall only consider conservative schemes.

In this paragraph we discuss questions of the convergence and accuracy of the conservative schemes $L_h^{(k,q,f)}$ in the classes $Q^{(m)}$ of discontinuous coefficients k, q, f of the differential equation.

We make use of the estimate

$$\| z^h \|_1 \leqslant M \| \varphi^h \|_2 ,$$

obtained in § 2, Section 3 (formula (30)) for the solution z_i^h of problem (III) with right-hand side φ_i^h. The norm

$$\| \varphi^h \|_2 = \sum_{i=1}^{N-1} h \Big| \sum_{s=1}^{i} h\varphi_s^h \Big|$$

proves to be a very effective means of proving not only theorems concerning the convergence and accuracy for discontinuous coefficients, but also of proving Theorem 3 concerning accuracy in the class $C^{(m)}$ of smooth coefficients, since by using this norm we can reduce the requirements both on the rank n of the scheme, and on the order m of the class of coefficients as compared with the corresponding theorem proved with the use of the inequality

$$\| z^h \|_1 \leqslant M \| \varphi^h \|_1, \qquad \| \varphi^h \|_1 = \max_{0<i<N} | \varphi_i^h |$$

(see § 2, Section 3, formula (29)).

1. *The convergence of conservative schemes in the class of discontinuous coefficients.*

THEOREM 1. *Any conservative scheme $L_h^{(k,q,f)}$ of zero rank converges in the class of piece-wise continuous coefficients $k, q, f \in Q^{(0)}$.*

For the sake of simplicity we give the proof for the case when there is only one point of discontinuity of the coefficients $k(x)$, $q(x)$ and $f(x)$, namely $\xi = xn + \theta h$ $(x_n = \theta h,\ 0 \leqslant \theta \leqslant 1)$. As usual, introducing the difference $z_i^h = y_i^h - u(x_i)$ we obtain for it the difference boundary problem

$$L_h^{(k,q)} z_i^h = -\varphi_i^h, \quad 0 < i < N, \quad z_0^h = 0, \quad z_N^h = 0, \tag{III}$$

where

$$\varphi_i^h = L_h^{(k,q,f)} u_i - (L^{(k,q,f)} u)_i .$$

We put the right-hand side φ_i^h in the form of the sum

$$\varphi_i^h = \bar{\varphi}_i^{(1)} + \bar{\bar{\varphi}}_i^{(1)} + \varphi_i^{(2)}, \tag{1}$$

where

$$\varphi_i^{(2)} = \varphi_n^h \delta_{in} + \varphi_{n+1}^h \cdot \delta_{i,n+1}, \quad \delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \tag{2}$$

$$\bar{\varphi}_i^{(1)} = -(D_i^{(h,q)} - q_i)u_i + (F_i^{(h,f)} - f_i), \quad i \neq n, \quad i \neq n+1, \tag{3}$$

$$\bar{\bar{\varphi}}_i^{(1)} = \frac{1}{h^2}\Delta(A_i^{(h,k)}\nabla u_i) - (ku')_i', \quad i \neq n, \quad i \neq n+1, \tag{4}$$

$$\bar{\varphi}_i^{(1)} = \bar{\bar{\varphi}}_i^{(1)} = 0, \quad i = n, \quad i = n+1.$$

If $i \neq n$, $i \neq n+1$ then

$$F_i^{(h,f)} = F^h[f(x_i + sh)] = F^{(0)}[f_i + \rho(h)] + \rho(h) = f_i + \rho(h) \tag{5}$$

and, similarly,

$$D_i^{(h,q)} = q_i + \rho(h), \tag{5'}$$

so that

$$\|\bar{\varphi}_i^{(1)}\|_1 = \rho(h). \tag{6}$$

In Section 1 of § 1 we remarked that if $k, q, f \in Q^{(0)}$ then the derivatives u', $(ku')'$ of the solution $u(x)$ of problem (I) are also piece-wise continuous. Therefore, for $i \neq n$ and $i \neq n+1$ we can write

$$(ku')_{i+1/2} = (ku')_i + \frac{h}{2}(ku')_i' + h\rho(h),$$

$$(ku')_{i-1/2} = (ku')_i - \frac{h}{2}(ku')_i' + h\rho(h),$$

and, therefore

$$(ku')_i' = \frac{(ku')_{i+1/2} - (ku')_{i-1/2}}{h} + \rho(h). \tag{8}$$

Since the derivative $u'(x)$ is continuous for $x \neq \xi$,

$$\frac{\nabla u_i}{h} = u'_{i-1/2} + \rho(h), \quad \text{if} \quad i \neq n+1. \tag{8}$$

As a result we can put the expression for $\bar{\varphi}_i^{(1)}$ in the form

$$\bar{\bar{\varphi}}_i^{(1)} = \frac{\Delta\Omega_i}{h} + \rho(h), \tag{9}$$

where

$$\Omega_i = (A_i^{(h,k)} - k_{i-1/2})u'_{i-1/2} + \rho(h). \tag{10}$$

Since $L_h^{(k)}$ is a scheme of zero rank,

$$A_i^{(h,k)} = k_i + \rho(h) = k_{i-1/2} + \rho(h), \quad i \neq n+1.$$

From this and form (10) we conclude that

$$\Omega_i = \rho(h) \quad \text{for all} \quad i \neq n+1. \tag{11}$$

To find the bounds of the solution of problem (III) we make use of the inequality obtained in § 2, Section 3:

$$\|z^h\|_1 \leqslant \overline{M}_5 \|\varphi^h\|_2. \tag{12}$$

Since the coefficients k and q satisfy the conditions

$$0 < M_1 \leqslant k(x) \leqslant M_2, \quad 0 \leqslant q(x) \leqslant M_3, \tag{α}$$

these conditions are also satisfied by $A_i^{(h,k)}$ and $D^{(h,q)}$. Then $\overline{M}_5 = \overline{M}_5(M_1, M_3)$ is a constant depending only on M_1 and M_3.

Putting $\varphi_i^h = \bar{\varphi}_i^{(1)} + \bar{\bar{\varphi}}^{(1)} + \varphi_i^{(2)}$ in (12) we obtain

$$\|\varphi^h\|_1 \leqslant M_5\{\|\bar{\varphi}^{(1)}\|_2 + \|\bar{\bar{\varphi}}^{(1)}\|_2 + \|\varphi^{(2)}\|_2\}.$$

Since $\|\varphi^h\|_2 \geqslant \|\varphi^h\|_1$, we have, from condition (6)

$$\|\bar{\varphi}^{(1)}\|_2 = \rho(h). \tag{13}$$

Further, we have

$$\|\bar{\bar{\varphi}}^{(1)}\|_2 = \sum_{i=1}^{n-1} h\,|\Omega_{i+1} - \Omega_1| + \sum_{i=N+2}^{N-1} h\,|(\Omega_{i+1} - \Omega_1) + (\Omega_n - \Omega_{n+2})| +$$

$$+ 2h\,|\Omega_n - \Omega_1| \leqslant 4 \max_{0<i<N(i\neq n+1)} |\Omega_i| + \rho(h). \tag{14}$$

Now using estimate (11) we have

$$\|\bar{\bar{\varphi}}^{(1)}\|_2 = \rho(h). \tag{15}$$

We calculate the norm of function $\varphi^{(2)}$ defined by formula (2):

$$\|\varphi^{(2)}\|_2 = h^2\,|\varphi_n^h| + h\,|\varphi_n^h + \varphi_{n+1}^h| \sum_{i=n+2}^{N-1} h \leqslant h^2\,|\varphi_n| + h|\varphi_n + \varphi_{n+1}|. \tag{16}$$

The necessary condition for convergence $\varphi_n + \varphi_{n+1} = \rho(h)$ is satisfied by our scheme, and $h\varphi_n = O(1)$; therefore

$$\|\varphi^{(2)}\|_2 = \rho(h). \tag{16$'$}$$

Collecting estimates (13), (15), and (16′) we obtain

$$\|z^h\|_1 = \rho(h). \tag{17}$$

2. *The accuracy of a conservative scheme.*

THEOREM 2. *The conservative scheme $L_h^{(k,q,f)}$ of the first rank has first order accuracy in the class of piece-wise continuous and piece-wise smooth functions k, q, $f \in Q^{(1)}$.*

To prove this theorem we need only repeat the proof of the Theorem 1, replacing $\rho(h)$ everywhere by $O(h)$.

THEOREM 3. *The conservative scheme $L_h^{(k,q,f)}$ of second rank, satisfying the conditions* (A.C.2) *of second order approximation, has second order accuracy in the class $C^{(2)}$ of coefficients k, q, f.*

However, such a scheme has, generally speaking, first order accuracy in $Q^{(m)}$ (for any $m \geqslant 1$).

If $k(x) \in C^{(2)}$ then

$$A_i^{(h,k)} = k_{i-1/2} + hk'_{i-1/2}(A_1^{(0)}[s]+0\cdot 5) + O(h^2) = k_{i-1/2} + O(h)^2, \tag{18}$$

since $A_1^{(0)}[s] = -0\cdot 5$ (the condition for second order approximation of the scheme $L_h^{(k)}$).

Since the derivatives u''' and k'' exist,

$$(ku')'_i = \frac{(ku')_{i+1/2} - (ku')'_{i-1/2}}{h} + O(h^2),$$

$$\Delta u_i/h = u'_{i+1/2} + O(h^2).$$

Put $\varphi_i^h = \varphi_i^{(1)} + \varphi_i^{(2)}$, where

$$\varphi_i^{(1)} = -(D_i^{(h,q)} - q_i)u_i + (F_i^{(h,f)} - f_i),$$

$$\varphi_i^{(2)} = \frac{\Delta(A_i^{(h,k)}\nabla u_i)}{h^2} - (ku')'_i = \frac{\Delta\Omega_i}{h} + O(h^2), \tag{19}$$

$$\Omega_i = (A_i^{(h,k)} - k_{i-1/2})u'_{i-1/2} + O(h^2).$$

Using condition (18) we obtain at once

$$\Omega_i = O(h^2). \tag{20}$$

From the conditions of the theorem it also follows that

$$D_i^{(h,q)} = q_i + O(h^2), \quad F_i^{(h,f)} = f_i + O(h^2), \quad \varphi_i^{(1)} = O(h^2). \tag{21}$$

Returning now to formula (12), for the solution of problem (III) we have

$$\| z^h \|_1 \leqslant M_5 \{\| \varphi^{(1)} \|_2 + \| \varphi^{(2)} \|_2\},$$

where $\| \varphi^{(1)} \|_2 = O(h^2)$ from condition (21).

Using formula (19)

$$\| \varphi^{(2)} \|_2 = \sum_{i=1}^{N} h|\Omega_i - \Omega_1| + O(h^2),$$

and with estimate (20) we obtain

$$\| \varphi^{(2)} \|_2 = O(h^2),$$

i.e. $\| z^h \|_1 = O(h^2)$ which we were required to prove.

COROLLARY. *The conservative symmetric scheme $L_h^{(k,q,f)}$ of second rank has second order accuracy in $C^{(2)}$.*

For such a scheme satisfies (A.C.2) i.e. all the conditions of Theorem 3.

NOTE. In Theorems 2 and 3 we can relax the requirement that the coefficients shall be differentiable. Theorem 3 remains true if $k, q, f \in C^{(1,1)}$ where $C^{(1,1)}$ is the class of functions whose first derivative satisfied the Lipschitz condition on the segment [0, 1]. Theorem 3 remains true if $k, q, f \in C^{(1,1)}$ where $C^{(1,1)}$ is the class of piece-wise continuous functions on the segment [0, 1], satisfying the Lipschitz condition in the intervals where they are continuous.

A similar remark applies to Theorem 4.

THEOREM 4. *For a conservative scheme of second rank satisfying the conditions for second order approximation to have second order accuracy in the class* $Q^{(m)}$ $(m \geqslant 2)$ *it is necessary and sufficient that it satisfies the conditions*

$$\varphi^h_{n_j} = O(1), \tag{a$_2$}$$

$$\varphi^h_{n_j} + \varphi^h_{n_j+1} = O(h) \tag{b$_2$}$$

in the neighbourhood of the points of discontinuity of the coefficients $k(x)$, $q(x)$ *and* $f(x)$, *the points* $\xi_j = x_{n_j} + \theta_j h$ $(x_{n_j}.h,\ 0 \leqslant \theta_j \leqslant 1,\ j = 1, 2, \dots, j_0)$.

We note that the difference between this theorem and the similar theorem of § 3 lies in the relaxation of the rank requirements for the scheme and of the order m of the class $Q^{(m)}$ of coefficients. The proof of necessity in this case too presents no difficulty. As usual we take the case of one discontinuity.

Dividing problem (III) into three problems, according to formulae (1)–(4) we see, by analogy with Section 1, that

$$\|\bar{\varphi}^{(1)}\|_1 = O(h^2),$$

$$\bar{\bar{\varphi}}_i^{(1)} = \frac{\Delta\Omega_i}{h} + O(h^2), \qquad \Omega_i = (A_i^{(h,k)} - k_{i-1/2})u'_{i-1/2} + O(h^2)$$

$$(i \neq n, \quad i \neq n+1),$$

$$\bar{\bar{\varphi}}_i^{(1)} = 0 \quad \text{for} \quad i = n, \quad n+1.$$

Since $L_h^{(k)}$ is a scheme of second rank, satisfying (A.C.2), we have $A_i^{(h,k)} = k_{i-1/2} + O(h^2)$ for $i \neq n+1$.

It follows that

$$\Omega_i = O(h^2) \quad \text{for} \quad i \neq n+1.$$

Now using

$$\|\bar{\varphi}^{(1)}\|_2 \leqslant 4 \overline{\max_{0<i<N,\, i\neq n+1}} |\Omega_i| + O(h^2),$$

by analogy with inequality (14), we have $\|\bar{\bar{\varphi}}^{(1)}\|_2 = O(h^2)$.

If $\bar{z}_i^{(1)}$ and $\bar{\bar{z}}_i^{(1)}$ are solutions of the problem (III) with right-hand sides $\bar{\varphi}_i^{(1)}$ and $\bar{\bar{\varphi}}_i^{(1)}$ respectively, then formula (12) and the estimates for $\|\bar{\varphi}^{(1)}\|_2$ and $\|\bar{\bar{\varphi}}^{(1)}\|_2$ give

$$\|\bar{z}^{(1)}\|_1 = O(h^2), \qquad \|\bar{\bar{z}}^{(1)}\|_1 = O(h^2). \tag{22}$$

Returning to the problem (III) for $z_i^{(2)} = z_i - (\bar{z}_i^{(1)} + \bar{\bar{z}}_i^{(1)})$ with the right-hand side $\varphi_i^{(2)} = \delta_{in}\varphi_n^h + \delta_{i,n+1}\varphi_{n+1}^h$ and using the results of § 3, Section 3, we see that the conditions (a$_2$) and (b$_2$) are necessary for $\|z^{(2)}\|_1$ to be equal to $O(h^2)$. From the inequalities

$$\|z^{(2)}\|_1 \leqslant \overline{M}_5 \|\varphi^{(2)}\|_2,$$

and

$$\|\varphi^{(2)}\|_2 \leqslant h^2|\varphi_n| + h|\varphi_n + \varphi_{n+1}| \tag{23}$$

we see that the conditions (a$_2$) and (b$_2$) are also sufficient for second order smallness of the function $z^{(2)}$ and (from (22)) the function z. This proves Theorem 4.

If there is a finite number of discontinuities at the points $\xi_j = x_{n_j} + \theta_j . h$ $(j = 1, 2, \ldots, j_0)$ instead of just one, then

$$\varphi_i^{(2)} = \sum_{j=1}^{j_0} \varphi_{ij}^{(2)},$$

where $\varphi_{ij}^{(2)} = \delta_{in_j}\varphi_{n_j}^h + \delta_{i,\, n_j+1}\varphi_{n_j+1}^h$, and instead of (16) and (23) we obtain the estimate

$$\| \varphi_i^{(2)} \|_2 \leqslant \sum_{j=1}^{j_0} [\, | \varphi_{n_j}^h | \, h^2 + | \varphi_{n_j}^h + \varphi_{n_j+1}^h | \, h],$$

since

$$\| \varphi_i^{(2)} \|_2 \leqslant \sum_{j=1}^{j_0} \| \varphi_{ij}^{(2)} \|_2 .$$

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